FREE PROBABILITY OF TYPE B: ANALYTIC INTERPRETATION AND APPLICATIONS

S.T. BELINSCHI⋆ AND D. SHLYAKHTENKO‡

Abstract. In this paper we give an analytic interpretation of free convolution of type B, and provide a new formula for its computation. This formula allows us to show that free additive convolution of type B is essentially a re-casting of conditionally free convolution. We put in evidence several aspects of this operation, the most significant being its apparition as an 'intertwiner' between derivation and free convolution of type A. We also show connections between several limit theorems in type A and type B free probability. Moreover, we show that the analytical picture fits very well with the idea of considering type B random variables as infinitesimal deformations to ordinary non-commutative random variables.

Free probability theory was introduced by D. Voiculescu in the eighties (see e.g. [23]). Already in his early work [22] Voiculescu has found analytical ways for computation of a number of natural operations in his theory, such as free convolution. Later on, Speicher [21] found that the combinatoris of free probability theory has to do with ("type A") non-crossing partitions. For example, he found a description of the relation between moments and cumulants in terms of the lattice $NC^{(A)}(n)$ of noncrossing partitions of $\{1,2,\ldots,n\}$; free independence could then be phrased in terms of vanishing of mixed cumulants. We refer the reader to [16] for a detailed description.

This paper is devoted to the exploration of a new notion of independence, called type B freeness, which was introduced by Biane, Goodman and Nica in [6]. The original motivation for the introduction of this notion was the fact that the lattice of non-crossing partitions, central to (ordinary, or "type A") probability theory is naturally associated to the symmetric groups, which are Weyl groups of Lie groups of type A. If the symmetric group is replaced by the hyperoctohedral group (the Weyl group of a type B Lie group), one obtains the lattice $NC^{(B)}(n)$ of type B non-crossing partitions [19], and thus one seeks a probabilty theory whose underlying combinatorics is governed by this other lattice. In [6], the authors have shown that type B free probability theory does indeed exist: one can make sense of a type B law and of a type B non-commutative random variable. There is a notion of freeness, which has the desired combinatorial structure. Finally, one has the notions of type B free convolutions of type B laws, together with an appropriate linearizing transform (the R-transform of type B,) and so on. Later in [18], M. Popa showed that there is a natural notion of type B semicircular law and a type B central limit theorem.

Type B freeness can be considered unusual in that there seemed to be no obvious notion of positivity. For a single random variable of type B, its law can be viewed as being described by a pair of measures (μ,μ'). Unfortunately, although it is clear that μ should be positive, there was no obvious positivity condition on μ'; and indeed, the measure μ' associated to a type B semicircular variable need not be positive (as remarked in [18] for the central limit and Poisson limit distributions). To find a reasonable positivity assumption, we chose to introduce a notion of *infinitesimal law* of a family of random variables, which is a weakening of the notion of a type B law (this notion is almost implicit in the work of Biane, Goodman and Nica; indeed, they show that type B probability is related to

Date: March 15, 2009

⋆ University of Saskatchewan and Institute of Mathematics of the Romanian Academy. 106 Wiggins Road, Saskatoon, Saskatchewan S7N 5E6, Canada. E-mail: belinschi@math.usask.ca. Supported by a Discovery grant from the Natural Sciences and Engineering Research Council of Canada and a University of Saskatchewan start-up grant.

‡ Department of Mathematics, UCLA, Los Angeles, CA 94720, U.S.A. E-mail: shlyakht@math.ucla.edu. Research supported by NSF grant DMS-0555680.

freeness with amalgamation over the algebra $\mathbb{C}+\mathbb{C}\hbar$ of power series in $\hbar$ taken modulo terms of order $\hbar^2$ or higher).

More precisely, there is an infinitesimal law associated to every family of type B random variables (though some informationis lost when passing to the infinitesimal law). This weakening, however, is of no consequence in a single-variable case, and amounts to interpreting the pair (μ,μ') as the zeroth and first derivative of a family of laws μ_t (i.e., $\mu=\mu_0$ and $\mu'(f)=\frac{d}{dt}\Big|_{t=0}\mu_t(f)$ for sufficiently nice f). One natural notion of positivity is then to require existence of a family μ_t of *positive* laws whose derivative is μ'. One can then check that the obvious notion of infinitesimal freeness (which requires that freeness conditions are fulfilled to order $o(t)$) are compatible with type B freeness. In particular, it turns out (Theorem 26) that free convolution of type B is intimately related to free convolution of type A: if $(\eta,\eta')=(\mu,\mu')\boxplus_B(\nu,\nu')$ then $\eta=\mu\boxplus\nu$ and $\eta'=\frac{d}{dt}(\mu_t\boxplus\nu_t)$ (usual free convolution), where μ_t and ν_t are any two families of laws having as their derivatives at $t=0$ μ' and ν', respectively.

Although the notions of infinitesimal law, infinitesimal freeness add to the proliferation of different notions of non-commutative random variables, freeness and so on, we feel that these notions are justified since they simplify our presentation and look rather natural. For example, the rather mysterious type B semicircular law is nothing by the infinitesimal law associated to the family of laws of variables $x+ty$ where x and y are two free semcirculars (Example 31).

The notion of infinitesimal freeness is in spirit related to the notion of second-order freeness introduced by Mingo and Speicher [14], but is different from it (our notion is related to a first derivative, and Speicher's is related to a second derivative, defined in the case the first derivative vanishes).

A very rich source of infinitesimal laws is given by random matrix theory. Indeed, if X_N is an $N\times N$ random matrix, its moments typically have an expansion in powers of $1/N$. Keeping the zeroth and first order terms in $1/N$ then gives rise to an infinitesimal law. Unfortunately, we were unable to find a direct connection between ordinary random matrices and type B freeness. The natural guess — taking X_N to be a *real* Gaussian random matrix and looking at its law to order $1/N$ as $N\to\infty$ does not produce an infinitesimal (i.e., type B) semicirclar variable. Indeed, as was shown by [11, 9], the law of an $N\times N$ real random matrix is approximately

$$\mu_t=\frac{2}{\pi\sqrt{1-t^2}}+t\left(\frac{1}{4}(\delta_1+\delta_{-1})-\frac{1}{2\pi\sqrt{1-t^2}}\right),\qquad t=N^{-1}.$$

On the other hand, the type B semicircular law is associated with the following infinitesimal law:

$$\mu_t=\mu_0+t(\mu_0-\nu),$$

where μ_0 is the semicircle law and ν is the arcsine law. Because of the type B free central limit theorem, it is clear that such independent real Gaussian matrices become in any way asymptotically B-free.

We show, however, that if we instead start with an $N\times N$ self-adjoint matrix X_N whose entries are semicircular variables, then the infinitesimal law associated to X_N by keeping expressions of order 0 and 1 in $1/N$ *does* converge to a type B semicircular variable (in fact, the law of X_N is semicircular of variance $(1+1/N)$ for all N). Note that X_N is "symmetric" in the sense that it is equal to the matrix obtained from X_N by transposing all rows and columns, and is thus a free probability analog of a real Gaussian random matrix (Corollaries 38 and 39.) The matrix X_N no longer posesses a unitary symmetry, but rather an orthogonal one. It would be interesting to investigate if a direct connection to the combinatorics of the hyperoctahedral group (related to the orthogonal Lie groups) can be made in this way.

One of the main goals of this paper is to give an analytic framework for type B free probability. Namely, we show that the operation $\boxplus_B$ of free additive convolution of type B is well defined on a product of two spaces, the first of which is the space of Borel probability measures on $\mathbb{R}$, and the second is essentially the space of distributions on the real line which are derivatives of positive finite measures, not necessarily probability measures. (We find in fact three such second coordinate

spaces which are stable under $\boxplus_B$.) In this context, it turns out that $\boxplus_B$ is a re-casting in terms of derivatives of Boolean cumulants of another operation, $\boxplus_C$, the conditionally free convolution introduced by Bożejko, Leinert and Speicher in [7]. We prove this result in Theorem 22, thus answering an open problem from [6]. Surprisingly, there is no counterpart of this correspondence for multiplicative convolutions.

The paper is organized in four sections. In Section 1 we introduce the main notions and tools used in our proofs. Section 2 establishes the connection between infinitesimal freeness and freeness of type B, for both single and multi-variable cases. Section 3 is dedicated to the description of free additive convolution of type B from an analytic perspective. In Section 4 we use results form Sections 2 and 3 to describe some limit measures, namely several type B stable distributions and the type B Poisson distribution. In Section 4 we provide a matricial model for type B freeness, and in Section 5 we discuss the operation $\boxtimes_B$ of free multiplicative convolution of type B.

Acknowledgements. We are indebted to Teodor Banica and Edouard Maurel-Segala for numerous useful discussions. STB would also like to thank Jacek Szmigielski for help in the choice of an appropriate space for the second coordinate of the free additive convolution of type B.

1. Notations and preliminary results

The key concept for our paper is presented in the following definition [6, Definition 6.1]:

Definition 1. *A noncommutative probability space of type B is a system $(\mathcal{A}, \tau, \mathcal{V}, \varphi, \Phi)$, where*

1. *$(\mathcal{A}, \tau)$ is a type A noncommutative probability space (i.e. $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and τ is a linear functional carrying the unit of the algebra into 1);*
2. *$\mathcal{V}$ is a complex vector space and $\varphi\colon \mathcal{V} \to \mathbb{C}$ is a linear functional;*
3. *$\Phi\colon \mathcal{A} \times \mathcal{V} \times \mathcal{A} \to \mathcal{V}$ is a two-sided action of $\mathcal{A}$ on $\mathcal{V}$. We will denote $\Phi(a, \xi, b)$ simply by $a\xi b$, for any $a, b \in \mathcal{A}, \xi \in \mathcal{V}$.*

For our purposes, we will need additional structure. Thus, from now on we will assume that $\mathcal{A}$ is a C^*-algebra, τ is positive, $\mathcal{V}$ is seminormed, φ is continuous, and the action Φ is separately continuous.

It was observed in [6] that one can define a structure of unital algebra on $\mathcal{A} \times \mathcal{V}$ as follows. We represent any vector $(a, \xi) \in \mathcal{A} \times \mathcal{V}$ as $\begin{bmatrix} a & \xi \\ 0 & a \end{bmatrix}$. Then

$$(a_1, \xi_1) \cdot (a_2, \xi_2) = \begin{bmatrix} a_1 & \xi_1 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & \xi_2 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & a_1\xi_2 + \xi_1a_2 \\ 0 & a_1a_2 \end{bmatrix} = (a_1a_2, a_1\xi_2 + \xi_1a_2).$$

The unit is simply $(1, 0)$, where 1 is the unit of $\mathcal{A}$. As observed in [18], we can view $\mathcal{A} \times \mathcal{V}$ as an operator-valued noncommutative probability space over the scalar (commutative) algebra

$$\mathcal{C} = \left\{ \begin{bmatrix} z & w \\ 0 & z \end{bmatrix} : z, w \in \mathbb{C} \right\} \subseteq \mathcal{M}_2(\mathbb{C}) \tag{1}$$

with the conditional expectation $E((a, \xi)) = (\tau(a), \varphi(\xi))$. (As observed in the introduction, this algebra is isomorphic to the algebra $\mathbb{C} + \hbar\mathbb{C}$ of power series taken modulo terms of order higher than two.) Since $\mathcal{C}$ is in the centre of $\mathcal{A} \times \mathcal{V}$, the notion of joint moments generalizes in the obvious way to the context of probability space over $\mathcal{C}$ (see also [21] and [18]). We will next define the non-crossing cumulants of type A and B.

Definition 2. *The type A free cumulants associated to a noncommutative probability space $(\mathcal{A}, \tau)$ are the family of multilinear functionals $\left(\kappa_n^{(A)}\colon \mathcal{A}^n \to \mathbb{C}\right)_{n=1}^{\infty}$ determined by the equation*

$$\sum_{\pi \in NC^{(A)}(n)} \prod_{F \in \pi} \kappa_{|F|}^{(A)}\left((a_1, \dots, a_n)|F\right) = \tau(a_1 \cdots a_n), \tag{2}$$

for all $a_1, \ldots, a_n \in \mathcal{A}$, $n \in \mathbb{N}$. *Here,* $F \in \pi$ *means "*F *is a block of* π*",* $|F|$ *denotes the cardinality of the block* F*, and if* $F = \{j_1 < j_2 < \cdots < j_{|F|}\}$*, the notation* $\kappa_{|F|}^{(A)}((a_1, \ldots, a_n)|F)$ *means* $\kappa_{|F|}^{(A)}(a_{j_1}, a_{j_2}, \ldots, a_{j_{|F|}})$.

We observe immediately that for a given k-tuple of random variables $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathcal{A}^k$, the free cumulants of type A of $\mathbf{a}$ represent simply a family of numbers unique determined by the joint moments of $\mathbf{a}$ and relation (2).

Following [6], we define the free cumulants of type B the following way:

Definition 3. *Let* $(\mathcal{A}, \tau, \mathcal{V}, \varphi, \Phi)$ *be a noncommutative probability space of type* B*. The type* B *free cumulants associated to it are the family of multilinear* $\mathcal{C}$*-valued functionals* $\left(\kappa_n^{(B)} : (\mathcal{A} \times \mathcal{V})^n \to \mathcal{C}\right)_{n=1}^{\infty}$*, uniquely determined by the equation*

$$\sum_{\pi \in NC^{(A)}(n)} \prod_{F \in \pi} \kappa_{|F|}^{(B)}((a_1, \xi_1), \ldots, (a_n, \xi_n)|F) = E((a_1, \xi_1) \cdots (a_n, \xi_n)), \tag{3}$$

where the notations correspond to the ones in Definition 2.

Thus, the free cumulants of type B are defined according to the same equation as the ones of type A, but viewed over the algebra $\mathcal{C}$. If one looks at the second coordinate of $\kappa_n^{(B)}((a_1, \xi_1), \ldots, (a_n, \xi_n))$, one sees that, while the first coordinate equals the type A free cumulant, the second is quite different.

The definition of freeness of type B coincides thus with the definition of freeness of type A for the first coordinate (i.e. for the pair $(\mathcal{A}, \tau)$) and the novelty element is brought by the second coordinate. More precisely,

Definition 4. *Let* $(\mathcal{A}, \tau, \mathcal{V}, \varphi, \Phi)$ *be noncommutative probability space of type* B*. Let* $\mathcal{A}_1, \ldots, \mathcal{A}_k$ *be unital subalgebras of* $\mathcal{A}$ *and* $\mathcal{V}_1, \ldots, \mathcal{V}_k$ *be linear subspaces of* $\mathcal{V}$ *so that* $\mathcal{V}_j$ *is invariant under the action of* $\mathcal{A}_j$*,* $1 \le j \le k$*. We say that* $(\mathcal{A}_1, \mathcal{V}_1), \ldots, (\mathcal{A}_k, \mathcal{V}_k)$ *are freely independent in* $(\mathcal{A}, \tau, \mathcal{V}, \varphi, \Phi)$ *if the following happens:*

(i) *For any* $n \in \mathbb{N}$*,* $i_1 \ne i_2 \ne \cdots \ne i_n \in \{1, \ldots, k\}$ *if* $a_j \in \mathcal{A}_{i_j}$ *satisfy* $\tau(a_j) = 0$ *for all* $1 \le j \le n$*, then* $\tau(a_1 \cdots a_n) = 0$ *(i.e.* $\mathcal{A}_1, \ldots, \mathcal{A}_k$ *are free in the type* A *sense in* $(\mathcal{A}, \tau)$*.)*

(ii) *The formula*

$$\varphi(a_m \cdots a_1 \xi b_1 \cdots b_n) = \begin{cases} 0 & \text{if } m \ne n \\ \delta_{i_1, j_1} \cdots \delta_{i_n, j_n} \tau(a_1 b_1) \cdots \tau(a_n b_n) \varphi(\xi) & \text{if } m = n \end{cases} \tag{4}$$

holds whenever

- $m, n \in \mathbb{N}$*, and* $i_m, \ldots, i_1, h, j_1 \ldots, j_n \in \{1, \ldots, k\}$ *are such that any two consecutive indices in the list are different from each other;*
- $a_m \in \mathcal{A}_{i_m}, \ldots, a_1 \in \mathcal{A}_{i_1}, \xi \in \mathcal{V}_h, b_1 \in \mathcal{A}_{j_1}, \ldots, b_n \in \mathcal{A}_{j_n}$ *are such that* $\tau(a_1) = \cdots = \tau(a_m) = 0 = \tau(b_1) = \cdots = \tau(b_n)$.

This definition appears as Definition 7.2 in [6]. Obviously, two type B random variables are free if they live in pairs (algebra, linear space) which are B-free.

Example 5. (We thank Benoît Collins for indicating this example to us.) A simple, and yet very useful, example can be obtained from type A freeness. Consider the type B probability space $(\mathcal{A}, \varphi, \mathcal{A}, \varphi, \Phi)$, where Φ is the action by multiplication from the left and by multiplication with elements from $\mathcal{A}^{\mathrm{op}}$ from the right, and $(\mathcal{A}, \varphi)$ is simply a type A noncommutative probability space (we will denote such a space just by $(\mathcal{A}, \varphi, \mathcal{A}, \varphi)$.) It follows easily from the definition of type A freeness that if $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are A-free in $(\mathcal{A}, \varphi)$, then $(\mathcal{A}_1, \mathcal{A}_1), \ldots, (\mathcal{A}_k, \mathcal{A}_k)$ are B-free in $(\mathcal{A}, \varphi, \mathcal{A}, \varphi)$. In particular, if $x, y \in \mathcal{A}$, then $(x, y) \in \mathcal{A} \times \mathcal{A}$ is a type B random variable.

For the purpose of studying free convolutions of type B, the following two observations are essential. First, in the world of *formal* power series with coefficients in $\mathcal{C}$, the type B R-transform

of a type B random variable linearizes type B free additive convolution: if $(a_1,\xi_1),(a_2,\xi_2)$ are free, then $R_{(a_1,\xi_1)}(z)+R_{(a_2,\xi_2)}(z)=R_{(a_1+a_2,\xi_1+\xi_2)}(z)$, where

$$R_{(a_j,\xi_j)}(z)=\sum_{n=1}^{\infty}\kappa_n^{(B)}(\underbrace{(a_j\xi_j),\dots,(a_j,\xi_j)}_{n\text{ times}})z^n.$$

Second, the type B combinatorial structure behaves identically to the type A structure viewed over the algebra $\mathcal{C}$. These results appear in [6], as Theorem 7.3 and Proposition 6.5.

An immediate consequence of these results is that all formal power series operations and properties used in type A free probability have a natural counterpart with the same properties in type B free probability when replacing complex numbers with elements from $\mathcal{C}$ as coefficients (more details below). This fact has been used massively in [6] in order to describe, among others, type B distributions of sums (see above) and products of of B-free random variables. Of course, these correspond to certain operations, defined on the space of type B distributions, which we will call free additive (respectively multiplicative) convolution of type B, and denote them $\boxplus_B$ and $\boxtimes_B$, respectively. In order to better capture the analytic structure of type B convolutions, it is helpful to view the operations on formal power series corresponding to $\boxplus_B$ and $\boxtimes_B$ as operations on analytic maps from $\mathcal{C}$ to itself. The following setup turns out to be the appropriate one: Define

$$Z=\begin{bmatrix} z & w\\ 0 & z\end{bmatrix}\in\mathcal{C}.$$

We immediately observe that

$$\begin{bmatrix} z & w\\ 0 & z\end{bmatrix}\begin{bmatrix} z' & w'\\ 0 & z'\end{bmatrix}=\begin{bmatrix} zz' & zw'+wz'\\ 0 & zz'\end{bmatrix},\quad z,z'w,w'\in\mathbb{C},$$

and thus

$$Z^n=\begin{bmatrix} z^n & nz^{n-1}w\\ 0 & z^n\end{bmatrix}\quad\forall n\in\mathbb{N},\quad Z^{-1}=\begin{bmatrix} z^{-1} & -wz^{-2}\\ 0 & z^{-1}\end{bmatrix}.$$

For any formal power series in one variable $\tilde{f}(z)=\sum_{n=0}^{\infty}A_nz^n$, with coefficients $A_n=\begin{bmatrix} a_n & b_n\\ 0 & a_n\end{bmatrix}\in\mathcal{C}$, as defined in [6, Section 5.2], we define the function

$$f(Z)=\sum_{n=0}^{\infty}A_nZ^n,\quad f\colon\Omega\subseteq\mathcal{C}\to\mathcal{C},$$

where Ω is some open set in $\mathcal{C}$. Such an f typically needs not make sense for any $Z\in\mathcal{C}$, so we must analyze carefully the two components of this complex map. We have $f=(f_1,f_2)=\begin{bmatrix} f_1 & f_2\\ 0 & f_1\end{bmatrix}$, where

$$f_1(z,w)=f_1(z)=\sum_{n=0}^{\infty}a_nz^n,\quad\text{and}\quad f_2(z,w)=\sum_{n=0}^{\infty}na_nz^{n-1}w+\sum_{n=0}^{\infty}b_nz^n=wf_1'(z)+g(z).$$

We observe immediately that when we compare this to the formal power series, we obtain

$$\tilde{f}(z)=\left(\sum_{n=0}^{\infty}a_nz^n,\sum_{n=0}^{\infty}b_nz^n\right)=(\tilde{f}_1(z),\tilde{g}(z)).$$

It is to be noted that only the first coordinate of Z is restricted by possible issues related to the radius of convergence of f_1,g. The natural power series operations obey this correspondence. The main advantage of using maps of $\mathcal{C}$ instead of formal power series is that it allows a notion of inverse map with respect to composition. Indeed, although the variable w seems to play a minor, rather inconvenient, role, in many circumstances it will allow us to define the *inverse with respect to composition* of a function f, noted f^{-1}, by the obvious condition $f\circ f^{-1}(Z)=Z$ and $f^{-1}\circ f(Z)=Z$.

We immediately observe that for f to be invertible we need f_1 as a one-variable function to be invertible and then, if $z = f_1^{-1}(\zeta)$ and $\Xi = (\zeta, v)$, we get

$$f(Z) = \Xi \iff f^{-1}(\Xi) = Z \iff z = f_1^{-1}(\zeta) \text{ and } w = \frac{v - g(f_1^{-1}(\zeta))}{f_1'(f_1^{-1}(\zeta))}.$$

Thus, whenever the one-variable complex function f_1 is well defined and (locally) invertible, and g is well defined, we can define the composition inverse f^{-1} of f.

Now let us observe that the formal power series $\tilde{f}$ is uniquely determined by the function f whenever f has a nonzero radius of convergence (meaning f_1 and g have, as one-variable complex analytic functions, non-zero radius of convergence each). Indeed, knowing f on any open set in $\mathcal{C}$ means knowing in particular f_1 and g in some open set in $\mathbb{C}$, and hence knowing the coefficients a_n, b_n for all $n \in \mathbb{N}$, i.e. knowing all A_n. Moreover, the set of such functions forms an algebra which is commutative and embeds in the algebra of formal power series with coefficients in $\mathcal{C}$. (Observe that the embedding is given by $(f_1, f_2) \mapsto (f_1(z), f_2(z, 0))$.) The unit is the constant function $e(z, w) = (1, 0)$.

We now use Theorem 16.15 in [16]:

Theorem 6. *For f, g formal power series in the s non-commuting variables $z_1, \dots, z_s$, without free term, with coefficients in a commutative algebra, the following two conditions are equivalent:*

$$\mathrm{Cf}_{(i_1,\dots,i_n)}(g) = \sum_{\pi \in NC^{(A)}(n)} \mathrm{Cf}_{(i_1\dots,i_n);\pi}(f) \quad \forall n \geq 1, 1 \leq i_1, \dots, i_n \leq s. \tag{5}$$

$$g = f(z_1(1+g), \dots, z_s(1+g)). \tag{6}$$

(Here $\mathrm{Cf}_{(i_1\dots,i_n);\pi}(f)$ denotes the product of coefficients of f indexed by the blocks of the partition π, where the coefficient corresponding to the block $\beta = \{j(1), \dots, j(r)\}$ is $\mathrm{Cf}_{(i_{j(1)}\dots,i_{j(r)})}(f)$, the coefficient of $z_{i_{j(1)}} \cdots z_{i_{j(r)}}$ in the expression of f.) We apply this theorem in the case $s = 1$ to formal power series with coefficients in $\mathcal{C}$ to obtain

Corollary 7. *The complex functions of $Z = \begin{bmatrix} z & w \\ 0 & z \end{bmatrix}$ defined by $M_{(a,\xi)}(Z) = \sum_{n=1}^{\infty} E((a,\xi)^n)Z^n$ and $R_{(a,\xi)}(Z) = \sum_{n=1}^{\infty} \kappa_n^{(B)}(\underbrace{(a,\xi), \dots, (a,\xi)}_{n})Z^n$ satisfy the usual moment-cumulant functional equation*

$$M_{(a,\xi)}(Z) = R_{(a,\xi)}(Z(\mathbf{1} + M_{(a,\xi)}(Z))),$$

where $\mathbf{1} = (1, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Moreover, the functions $M_{(a,\xi)}$ and $R_{(a,\xi)}$ determine each other uniquely via the above functional equation and the condition that their degree zero term is zero.

Remark 8. It will be more convenient for us to work with the appropriate generalization of the Cauchy transform:

$$G_{(a,\xi)}(Z) = \sum_{n=0}^{\infty} A_n Z^{-n-1} = \left(\sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}, w \sum_{n=0}^{\infty} \frac{-(n+1)a_n}{z^{n+2}} + \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}} \right),$$

where $A_0 = (1, 0)$, $a_j = \tau(a^j)$, $b_j = \sum_{k=1}^{j} \varphi(a^{k-1}\xi a^{j-k})$. We will denote the first component by $G_a(z)$ and the function $\sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}$ by $g_\xi(z)$ (or, when they will be viewed as corresponding to a distribution pair (μ, ν) having the corresponding moments (a_n, b_n), by $G_\mu(z)$ and $g_\nu(z)$).

It will be seen later that, while for the first coordinate of a type B distribution the appropriate object is indeed a Borel probability measure on the real line, there are several appropriate possible choices for the second. The definition below will provide the largest convenient framework, from which we will particularize as needed.

Definition 9. *Denote $\mathcal{M}$ the set of all Borel probability measures on $\mathbb{R}$ and $\mathcal{M}_0$ the set of linear functionals ν defined on the complex vector space generated by the functions $\mathbb{R} \ni t \mapsto (z-t)^{-n}$, $n \geq 0$, $\Im z \neq 0$, satisfying the following conditions: (i) $\nu(1) = 0$, (ii) the correspondence $z \mapsto \nu((z-t)^{-n})$ is analytic and satisfies (a) $\nu((\bar{z}-t)^{-n}) = \overline{\nu((z-t)^{-n})}$, and (b) $\lim_{y \to +\infty}(iy)^n \nu((iy-t)^{-n}) = 0$. We will define the support of ν to be the complement of the (open) subset of $\mathbb{C} \cup \{\infty\}$ on which $z \mapsto \nu((z-t)^{-n})$ has a unique analytical extension satisfying (a).*

(As an example, Schwarz distributions with compact support on $\mathbb{R}$ satisfying condition (i), satisfy also (ii).) Also, denote $\mathbb{C}^+ = \{z \in \mathbb{C} \colon \Im z > 0\}$, the upper half-plane of the complex plane.

There are several justifications for defining $\mathcal{M}_0$ this way, in terms of the functions $t \mapsto (t-z)^{-n}$ (beyond the obvious reason that "it works".) As the reader might recall from the introduction, one expects the second coordinate of a type B distribution to be in a certain sense a derivative of a probability measure. The theory of distributions allows one to extend the notion of derivative to more general objects than differentiable functions, roughly by using integration by parts. As a relevant example, for a compactly supported finite measure ν on $\mathbb{R}$, if f is a smooth enough function, then $\int f' \, d\nu = -\int f \, d\nu'$. This kind of derivative is known as *distributional derivative.* (The reader can find more information about the origins of this very rich subject in [20].) On the other hand, possibly up to a sign, the derivative of $(t-z)^{-n}$ with respect to z coincides with the derivative with respect to t. Thus, "philosophically" one can view in the above definition the requirement that ν is defined on all functions $t \mapsto (t-z)^{-n}$ together with (b), (ii) as knowing (and allowing) the derivatives of all orders of ν, condition (a), (ii) as requiring that ν is in a certain sense real, and condition (i) as conveniently generalizing the demand that ν is a derivative of a probability on $\mathbb{R}$. Requiring that ν is defined on functions $t \mapsto (t-z)^{-n}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$ corresponds to requiring that ν be supported on the real line. When ν is the distributional derivative of a Borel probability measure on $\mathbb{R}$, the above statements are precisely true.

Also, finite measures ν on the real line are fully recoverable from the Cauchy transform $z \mapsto \int (t-z)^{-1} \, d\nu(t)$, $\Im z \neq 0$, in terms of the nontangential limit of this function at points of the real line. This fact has been used with considerable success before in free probability, for ex. in [22, 4, 24, 3] etc. (For a very rich and detailed discussion of this problem, we refer the reader to the classical text of Akhieser [1].) However, it is not only finite measures on $\mathbb{R}$ that are described by their actions on functions $t \mapsto (t-z)^{-1}$. While we do not plan to pursue the subject here, we will mention that linear functionals defined on spaces of functions with L^p derivatives can also be recovered from the nontangential limits at points of $\mathbb{R}$ of their evaluation on $t \mapsto (t-z)^{-1}$, as shown in [13].

Remark 10. Let us observe that, when we consider distributions which are compactly supported in $\mathbb{R}$, these objects will be defined, and completely described by their action, on monomials t^n, $n \in \mathbb{N}$. Indeed, this fact is well-known for measures (see, for example, [1, Theorem 2.6.4]). For objects $\nu \in \mathcal{M}_0$, we will show first that one can define the values $\nu(t^n)$, and then that these values determine uniquely the functions $z \mapsto \nu((z-t)^{-n})$. Indeed, since $(t-z)^{-1} = \sum_{n=0}^{\infty} t^n z^{-n-1}$ for $|z| > |t|$, to find $\nu(t)$ one can, using (i) and the standard methods provided in [1, Chapter 3], start by formally writing

$$\nu(t) = \lim_{z \to \infty} z^2 \nu((z-t)^{-1}), \quad \nu(t^n) = \lim_{z \to \infty} z^{n+1} \left[\nu((z-t)^{-1}) - \sum_{j=0}^{n-1} \nu(t^j) z^{-j-1} \right].$$

Now recalling the hypothesis of compact support for ν, it follows that $z \mapsto \nu((z-t)^{-1})$ is analytic on a neighbourhood of infinity. From this hypothesis, together with the linearity of ν, it follows that all the numbers $\nu(t^n)$, $n \geq 0$, are well-defined, and moreover, they uniquely specify $\nu((z-t)^{-n})$ for all $n \geq 0$. Thus, provided it converges on some neighbourhood of infinity, the series $g_\nu(z) = \sum_{n=0}^{\infty} \nu(t^n) z^{-n-1}$ uniquely determines ν.

Next we shall recall some useful facts related to Cauchy transforms of free additive convolutions of probability measures on the real line. These results have been proved first by Voiculescu in [24] under some mild restrictions, and later in full generality by Biane in [5].

Theorem 11. *For any measure λ, let G_λ be its Cauchy transform and $F_\lambda = \frac{1}{G_\lambda}$. Assume $\mu_1, \mu_2 \in \mathcal{M}$ and denote $\mu_3 = \mu_1 \boxplus \mu_2$. Then there exist two analytic maps $\omega_1, \omega_2 \colon \mathbb{C}^+ \to \mathbb{C}^+$, uniquely determined by the following properties:*

1. $G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_2(z)) = G_{\mu_3}(z)$, $z \in \mathbb{C}^+$;
2. $\omega_1(z) + \omega_2(z) = z + F_{\mu_3}(z)$, $z \in \mathbb{C}^+$;
3. *If μ_1, μ_2 have compact support, then ω_j are analytic on some neighbourhood of infinity, $j \in \{1,2\}$;*
4. $\lim_{y\to+\infty} \omega_j(iy)/iy = \lim_{y\to+\infty} \omega_j'(iy) = 1$, $j \in \{1,2\}$.

The above relations determine uniquely the free additive convolution of μ_1 and μ_2.

Thus, G_{μ_3} is subordinated to G_{μ_1} and G_{μ_2} in the sense of Littlewood (see [10, Section 1.5]).

2. Type B free probability vs infinitesimal free probability theory.

2.1. **Infinitesimal laws of non-commutative random variables.** Let $X_t^{(j)}$ be a family of non-commutative random variables in a non-commutative probability space $(\mathcal{A}, \tau)$. Here $j = 1, \ldots, n$ and t is a parameter lying in the set $K \subset \mathbb{R}$ having zero as an accumulation point. We now define an object that captures the behavior of the moments of the family $\{X_t^{(j)} : t \in K\}$ as $t \to 0$ to order t (ignoring orders higher than t).

Definition 12. *An* infinitesimal law *of n variables is a pair of linear functionals $\mu, \mu' : \mathbb{C}\langle t_1, \ldots, t_n\rangle \to \mathbb{C}$ defined on the algebra of non-commutative polynomials in n indeterminates. We require that $\mu(1) = 1$ and $\mu'(1) = 0$.*

The intuitive idea is that if we have a family $X_t^{(j)}$ as above, we would set, for $p \in \mathbb{C}\langle t_1, \ldots, t_n\rangle$

$$\begin{aligned}
\mu(p(t_1, \ldots, t_n)) &= \lim_{t\to 0} \tau(p(X_t^{(1)}, \ldots, X_t^{(n)})), \\
\mu'(p(t_1, \ldots, t_n)) &= \lim_{t\to 0} \frac{1}{t}\left[\tau(p(X_t^{(1)}, \ldots, X_t^{(n)})) - \mu(p(t_1, \ldots, t_n))\right].
\end{aligned}$$

These could also be written as

$$\begin{aligned}
\mu &= \lim_{t\to 0} \mu_t \\
\mu' &= \lim_{t\to 0} \frac{1}{t}(\mu_t - \mu),
\end{aligned}$$

where μ_t denotes the law of the n-tuple $(X_t^{(j)} : j = 1, \ldots, n)$.

Given μ, μ', one can define a natural one-parameter family of laws having the infinitesimal law (μ, μ'), namely, $\mu_t = \mu + t\mu'$ (note that this is a family of laws since $\mu'(1) = 0$ and thus $\mu_t(1) = 1$ for all t).

2.2. **Freeness for infinitesimal laws.** There is also an obvious notion of freeness. Namely, we say that two families $X_t^{(j,1)}$, $j = 1, \ldots, n_1$ and $X_t^{(j,2)}$, $j = 1, \ldots, n_2$ are free (to order t) if one has, for arbitrary polynomials $P_1, \ldots, P_k$

$$\tau\left(\left[P_1(\vec{X}_t^{(i_1)}) - \tau(P_1(\vec{X}_t^{(i_1)}))\right] \cdots \left[P_k(\vec{X}_t^{(i_k)}) - \tau(P_k(\vec{X}_t^{(i_k)}))\right]\right) = o(t) \qquad \text{as } t \to 0$$

whenever $i_1, \ldots, i_k \in \{1,2\}$, $i_1 \neq i_2$, $i_2 \neq i_3$, $\ldots$, and we have set $\vec{X}_t^{(i)} = (X_t^{(1,i)}, \ldots, X_t^{(n_i,i)})$.

Thus we make the following definition:

Definition 13. *Let (μ,μ') be an infinitesimal law defined on $n+m$ variables $t_1,\dots,t_{n+m}$. We say that $(t_1,\dots,t_n)$ and $(t_{n+1},\dots,t_{n+m})$ are* infinitesimally free with respect to (μ,μ') *if $(t_1,\dots,t_n)$ and $(t_{n+1},\dots,t_{n+m})$ are freely independent to order t with respect to the law $\mu+t\mu'$.*

It is not hard to see that this condition translates into two requirements: (i) that $(t_1,\dots,t_n)$ and $(t_{n+1},\dots,t_{n+m})$ be freely independent with respect to the law μ and (ii) that

$$\mu'\left([p_1-\mu(p_1)]\cdots[p_k-\mu(p_k)]\right)-\sum_j \mu\left([p_1-\mu(p_1)]\cdots[\mu'(p_j)]\cdots[p_k-\mu(p_k)]\right)=0 \tag{7}$$

for any polynomials $p_1,\dots,p_k$ so that $p_j\in A_{i(j)}$, $i(1)\neq i(2)$, $i(2)\neq i(3)$, $\dots$, where $A_1=\mathbb{C}\langle t_1,\dots,t_n\rangle$ and $A_2=\mathbb{C}\langle t_{n+1},\dots,t_m\rangle$.

Proposition 14. *Assume that $A_1=\mathbb{C}\langle t_1,\dots,t_n\rangle$ and $A_2=\mathbb{C}\langle t_{n+1},\dots,t_m\rangle$ are infinitesimally free in $\mathbb{C}\langle t_1,\dots,t_{n+m}\rangle$ with respect to the law (μ,μ'). Then the restriction of (μ,μ') to the subalgberas A_1 and A_2 determines (μ,μ').*

Proof. Indeed, this is the case for μ (because of freeness); and (7) together with linearity and the requirement that $\mu'(1)=0$ defines μ' in terms of its restriction to A_1 and A_2. $\square$

Remark 15. *It is immediate from the definition that if two families $X_t^{(j,1)}$, $j=1,\dots,n_1$ and $X_t^{(j,2)}$, $j=1,\dots,n_2$ are actually freely indepedent (for all t), then they are also infinitesimally free.*

In particular, if we are given two infinitesimal laws (μ,μ'), (ν,ν'), and we set $\mu_t=\mu+t\mu'$, $\nu_t=\nu+t\nu'$, then

$$\begin{aligned}\eta &= \mu*\nu\\ \eta' &= \frac{d}{dt}\Big|_{t=0}(\mu_t*\nu_t)\end{aligned}$$

(here $\mu*\nu$ denotes the free product of two laws) gives rise to an infinitesimal law (η,η') of $n+m$ variables so that its restrictions to the first n and the last m variables are exactly (μ,μ') and (ν,ν'). We denote this law by $(\eta,\eta')=(\mu,\mu')*(\nu,\nu')$. This is the (unique because of Proposition (14)) free product of (μ,μ') and (ν,ν').

2.3. **Connection with type B free independence.** Let now $(X^{(j)},\xi^{(j)})$ be type B random variables in a non-commutative type B probability space $(A,\tau,\mathcal{V},f,\Phi)$. Associated to them we consider an infinitesimal famly of (type A) random variables

$$X_\hbar^{(j)}=\begin{bmatrix} X^{(j)} & \xi^{(j)}\\ 0 & X^{(j)}\end{bmatrix}=X^{(j)}+\hbar\xi^{(j)},$$

where $\hbar$ is a formal variable satisfying $\hbar^2=0$. In other words, we consider the infinitesimal law (μ_0,μ_0') given by

$$\begin{aligned}\mu_0(t_{i_1},\dots,t_{i_n}) &= \tau(X^{(i_1)}\cdots X^{i_n})\\ \mu_0'(t_{i_1},\dots,t_{i_n}) &= \sum_j f(X^{(i_1)}\cdots X^{(i_{j-1})}\xi^{(i_j)}X^{(i_{j+1})}\cdots X^{(i_n)}).\end{aligned}$$

We shall call this *the infinitesimal law* associated to the type B family $(X^{(j)},\xi^{(j)})$.

Note that (μ_0,μ_0') does not capture all of the type B law of the original family but only certain averages of moments (it does, however, capture the "type A part" of the law of $(X^{(j)},\xi^{(j)})$, which is exactly μ_0). For example,

$$\mu_0'(X_1X_2X_1X_2)=f(X_1\xi_2X_1X_2)+f(\xi_1X_2X_1X_2)+f(X_1X_2\xi_1X_2)+f(X_1X_2X_1\xi_2).$$

Even if we assume some tracialy of f, e.g. $f(A\xi_jB)=f(BA\xi_j)=f(\xi_jBA)$, the four terms on the right reduce to two terms $2(f(X_1X_2X_1\xi_2)+f(X_2X_1X_2\xi_1))$ but the equation still cannot be used to determine fully the type B law of the family (X_j,ξ_j).

Nonetheless we have:

Proposition 16. *If $((X^{(1)},\xi^{(1)}),\dots,(X^{(n)},\xi^{(n)})$ and $((X^{(n+1)},\xi^{(n+1)}),\dots,(X^{(n+m)},\xi^{(n+m)}))$ are two families of type B variables in $(A,\mathcal{V},\tau,f,\Phi)$ which are free, then the infinitesimal law (η,η') associated to $((X^{(1)},\xi^{(1)}),\dots,(X^{(n+m)},\xi^{(n+m)}))$ is the free product of the infinitesimal laws (μ,μ') and (ν,ν') associated to the families $((X^{(1)},\xi^{(1)}),\dots,(X^{(n)},\xi^{(n)}))$ and $((X^{(n+1)},\xi^{(n+1)}),\dots,(X^{(n+m)},\xi^{(n+m)}))$.*

Proof. We first note that $\eta=\mu*\nu$, because type B freeness entails (type A) freeness of the families $(X^{(1)},\dots,X^{(n)})$ and $(X^{(n+1)},\dots,X^{(n+m)})$.

Next, define a derivation $D:A\to\mathcal{V}$ given on monomials by

$$D(X^{(i_1)}\cdots X^{(i_k)})=\sum_j X^{(i_1)}\cdots X^{(i_{j-1})}\xi^{(i_j)}X^{(i_{j+1})}\cdots X^{(i_n)}.$$

Thus by definition $\mu'=f\circ D$.

Now, for any polynomials $p_1,\dots,p_k$ so that $p_j\in A_{i(j)}$, $i(1)\neq i(2)$, $i(2)\neq i(3)$,$\dots$, where $A_1=\mathbb{C}\langle t_1,\dots,t_n\rangle$ and $A_2=\mathbb{C}\langle t_{n+1},\dots,t_m\rangle$, we compute

$$\begin{aligned}\mu'\left([p_1-\mu(p_1)]\cdots[p_k-\mu(p_k)]\right)-\sum_j\mu\left([p_1-\mu(p_1)]\cdots[\mu'(p_j)]\cdots[p_k-\mu(p_k)]\right)\\ =\sum_j f\left([p_1-\tau(p_1)]\cdots Dp_j\cdots[p_k-\tau(p_k)]\right)\\ -\sum_j\tau\left([p_1-\tau(p_1)]\cdots[f(Dp_j)]\cdots[p_k-\tau(p_k)]\right)=0\end{aligned}$$

because the sums cancel term-by-term owing to type B freeness. □

2.3.1. *Single variable case.* In the case that we have a single type B random variable X in a type B non-commutative probability space $(A,\mathcal{V},\tau,f,\Phi)$ satisfying a traciality condition, the infinitesimal law associated to X determines its type B distribution.

In the single variable case, τ is a trace. We will make the assumption that the linear map f satisfies

$$f(X^k\xi X^l)=f(X^{k+l}\xi)$$

for all k,l. In this case, the infinitesimal family $\mu_\hbar=\mu_0+\hbar\mu_0'$ determines f and τ completely by the formulas:

$$\tau(X^n)=\int t^n d\mu_0(t),\qquad f(X^{j-i}\xi X^i)=\frac{1}{j+1}\int t^{j+1}d\mu_0'(t).$$

2.4. **Free additive convolution for infinitesimal laws.** Given two infinitesimal laws (μ,μ') and (ν,ν') defined on algebras $\mathbb{C}[t_1]$ and $\mathbb{C}[t_2]$ we define their infinitesimal free addtive convolution by

$$(\mu,\mu')\boxplus(\nu,\nu')=((\mu,\mu')*(\nu,\nu'))|_{\mathrm{Alg}(t_1+t_2)}.$$

In other words, the additive free convolution is the push-forward (under the addition map $(t_1,t_2)\mapsto t_1+t_2$) of their free product.

We note that there are two ways to compute this, in view of Proposition 16 and Remark 15:

Proposition 17. *Let $(\eta,\eta')=(\mu,\mu')\boxplus(\nu,\nu')$. Then:*
(a) $\eta=\mu\boxplus\nu$ (ordinary type A free convolution) and if we set $\mu_t=\mu+t\mu'$, $\nu_t=\nu+t\nu'$, then $\eta'=\frac{d}{dt}\Big|_{t=0}\mu_t\boxplus\nu_t$;
(b) Let μ^B and ν^B be the type B laws associated to μ and ν as in §2.3.1. Then (η,η') is the infinitesimal law associated to $\mu^B\boxplus_B\nu^B$.

3. Analytic computation of free additive convolution of type B

3.1. **Type B free additive convolution.** We observe [6] that if (a_1,ξ_1) and (a_2,ξ_2) are B-free, then the distribution of their sum depends only on the distributions of the two summands. Thus, it is possible to define a type B free additive convolution, as an operation on the space of sequences of pairs of complex numbers (a_n,b_n), $n\in\mathbb{N}$, by using the moment-cumulant formula given in Definition 3 and the linearizing property of the type B free cumulants. We will denote this operation by $\boxplus_B$. However, as in the case of the free convolution of type A, one would like to find the appropriate analytic object which will be stable under $\boxplus_B$. There are several relevant answers to this question. We shall first describe in the proposition below the analytic interpretation for the operation $\boxplus_B$ as described in [6]. (For the notation $\mathcal{M}_0$ used below we refer the reader to Definition 9.)

Proposition 18. *Consider two type B random variables $(a_1,\xi_1),(a_2,\xi_2)$ which are B-free, and are distributed according to (μ_1,ν_1) and (μ_2,ν_2), respectively. Assume that $\mu_1,\mu_2\in\mathcal{M}$ and $\nu_1,\nu_2\in\mathcal{M}_0$ are compactly supported on $\mathbb{R}$. Denote by (μ_3,ν_3) the distribution of $(a_1+a_2,\xi_1+\xi_2)$. Then, with the notations from Theorem 11 and Remark 8, we have*

(a) $\mu_3=\mu_1\boxplus\mu_2$;
(b) $g_{\nu_3}(z)=g_{\nu_1}(\omega_1(z))\omega_1'(z)+g_{\nu_2}(\omega_2(z))\omega_2'(z)$, $z\in\mathbb{C}^+$.

Moreover, $\mu_3\in\mathcal{M}$, $\nu_3\in\mathcal{M}_0$ have compact support in $\mathbb{R}$.

Proof. It will follow from the Corollary 7 and the corresponding type A theory that one can find two subordination functions Ω_1,Ω_2 so that

$$\Omega_1(Z)+\Omega_2(Z)=Z+F_{(a_1+a_2,\xi_1+\xi_2)}(Z)\quad\text{and}\quad G_{(a_j,\xi_j)}(\Omega_j(Z))=G_{(a_1+a_2,\xi_1+\xi_2)}(Z),j\in\{1,2\};\tag{8}$$

(recall that we denote by F the multiplicative inverse of G). Indeed, we will prove this fact by directly finding a formula for Ω_j; the proof will provide simultaneously the part (b) of our proposition. Let us first observe that part (a) follows directly from [6, Section 7.2]. Moreover, from (a), Theorem 11, and Remark 8, it follows that, if existing, the first coordinate of Ω_j depends only on the first coordinate of Z and coincides with the subordination function ω_j provided by Theorem 11. So denote $\Omega_j(Z)=\Omega_j(z,w)=(\omega_j(z),o_j(z,w))$. The subordination relation requires then for the second coordinate that

$$wG'_{\mu_3}(z)+g_{\nu_3}(z)=o_j(z,w)G'_{\mu_j}(\omega_j(z))+g_{\nu_j}(\omega_j(z)).\tag{9}$$

The second coordinate of the first relation in (8) (the analogue of Theorem 11 (b)) gives

$$o_1(z,w)+o_2(z,w)=w-\frac{wG'_{\mu_3}(z)+g_{\nu_3}(z)}{G_{\mu_3}(z)^2}=w(1+F'_{\mu_3}(z))-\frac{g_{\nu_3}(z)}{G_{\mu_3}(z)^2}.$$

We shall isolate from the above two equations o_1. Indeed, it follows easily that $o_2(z,w)=[wG'_{\mu_3}(z)+g_{\nu_3}(z)-g_{\nu_2}(\omega_2(z))]\cdot[G'_{\mu_2}(\omega_2(z))]^{-1}$. Amplifying the right-hand term by $\omega_2'(z)$ and then replacing in the equation above yields

$$o_1(z,w)+\frac{wG'_{\mu_3}(z)\omega_2'(z)+g_{\nu_3}(z)\omega_2'(z)-g_{\nu_2}(\omega_2(z))\omega_2'(z)}{G'_{\mu_2}(\omega_2(z))\omega_2'(z)}=w(1+F'_{\mu_3}(z))-\frac{g_{\nu_3}(z)}{G_{\mu_3}(z)^2},$$

and thus, by the chain rule and Theorem 11,

$$\begin{aligned}o_1(z,w)&=w(1+F'_{\mu_3}(z)-\omega_2'(z))+g_{\nu_3}(z)\cdot\frac{F'_{\mu_3}(z)-\omega_2'(z)}{G'_{\mu_3}(z)}+\frac{g_{\nu_2}(\omega_2(z))\omega_2'(z)}{G'_{\mu_3}(z)}\\&=w\omega_1'(z)+\frac{g_{\nu_3}(z)(\omega_1'(z)-1)+g_{\nu_2}(\omega_2(z))\omega_2'(z)}{G'_{\mu_3}(z)}.\end{aligned}\tag{10}$$

This provides the complete formula for the subordination function Ω_1. To conclude the proof of part (b) of the proposition, one needs only to replace the above formula for o_1 in (9), for

$j = 1$:

$$wG'_{\mu_3}(z)+g_{\nu_3}(z) = wG'_{\mu_1}(\omega_1(z))\omega'_1(z)+g_{\nu_1}(\omega_1(z))+G'_{\mu_1}(\omega_1(z))\cdot\frac{(\omega'_1(z)-1)g_{\nu_3}(z)+g_{\nu_2}(\omega_2(z))\omega'_2(z)}{G'_{\mu_3}(z)};$$

multiplication by $\omega'_1(z)$ and an application of Theorem 11 yields the desired formula

$$g_{\nu_3}(z) = g_{\nu_1}(\omega_1(z))\omega'_1(z) + g_{\nu_2}(\omega_2(z))\omega'_2(z).$$

The compacity of the support of μ_3 is known. It follows from the compacity of the supports of ν_1, ν_2, the above formula, and Theorem 11 that g_{ν_3} is analytic on a neighbourhood of infinity, that $g_{\nu_3}(\bar{z}) = \overline{g_{\nu_3}(z)}$ and that $\lim_{z\to\infty} zg_{\nu_3}(z) = 0$. Expanding g_{ν_3} in power series around infinity provides the values $\nu_3(t^n)$, $n \geq 0$, as seen in Remark 10, and makes possible to define ν_3 as a linear functional on the space of functions $t \mapsto (z-t)^{-n}$ in the obvious way. The property (ii), (b), of Definition 9 follows easily from the above and is left as an exercise. This concludes the proof of our proposition. □

3.2. **A connection with conditionally free convolution.** We follow next with a rather surprising application of the result above, namely we show that the type B free additive convolution encodes, up to translation, the conditionally free convolution (abbreviated c-free convolution) $\boxplus_C$ introduced by Bożejko, Leinert and Speicher in [7]. This operation is defined on pairs of probability measures on the real line $(\mu, \rho) \in \mathcal{M} \times \mathcal{M}$, and on the first coordinate acts as the free additive convolution: if $(\mu_1, \rho_1), (\mu_2, \rho_2) \in \mathcal{M} \times \mathcal{M}$ and we denote $(\mu_3, \rho_3) = (\mu_1, \rho_1) \boxplus_C (\mu_2, \rho_2)$, then $\mu_3 = \mu_1 \boxplus \mu_2$. In particular, the Cauchy transforms for the first coordinate[1] satisfy Theorem 11.

For any $\lambda \in \mathcal{M}$ we shall denote $h_\lambda(z) = F_\lambda(z) - z, z \in \mathbb{C}^+$. It is a consequence of [1, Equation 3.3] that h_λ takes values in the closure of the upper half-plane, and is real if and only if λ is a point mass.

Remark 19. The following representation for h_λ, called the Nevanlinna representation, will be used in stating and proving our next result: for any $\lambda \in \mathcal{M}$ there exist $a \in \mathbb{R}$ and a positive finite Borel measure σ on the real line so that

$$h_\lambda(z) = a + \int_{\mathbb{R}} \frac{1+tz}{t-z}\, d\sigma(t), \quad z \in \mathbb{C}^+.$$

Observing that $\frac{1+tz}{t-z} = (1+t^2)\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)$, it follows that one can write

$$h_\lambda(z) = \underbrace{a - \int_{\mathbb{R}} \frac{t}{1+t^2}(1+t^2)d\sigma(t)}_{\tilde{a}} + \int_{\mathbb{R}} \frac{1}{t-z} \underbrace{(1+t^2)d\sigma(t)}_{\tilde{\sigma}} = \tilde{a} - G_{\tilde{\sigma}}(z),$$

provided that σ has finite second moment (i.e. $\int_{\mathbb{R}} t^2\, d\sigma(t) < +\infty$.) It is shown in [1, Chapter 3] that this happens whenever λ has finite second moment (for the convenience of the reader, we will provide a sketch of the proof in the lemma below.) Assume for now that this condition holds. Then we can define an object $\tilde{\sigma}' \in \mathcal{M}_0$ by the relation $G_{\tilde{\sigma}'}(z) = -G'_{\tilde{\sigma}}(z)$, i.e.

$$G_{\tilde{\sigma}'}(z) = -G'_{\tilde{\sigma}}(z) = h'_\lambda(z), \quad z \in \mathbb{C}^+.$$

In particular, we observe that $\tilde{\sigma}'$ is the distributional derivative of $\tilde{\sigma}$, and in particular it indeed belongs to $\mathcal{M}_0$. For convenience, we shall denote by $\mathcal{M}_d$ the space of distributional derivatives of positive finite measures on the real line.

Lemma 20. *For any $\sigma \in \mathcal{M}_d$ there exists a unique $\rho \in \mathcal{M}$ so that $\int_{\mathbb{R}} t^2\, d\rho(t) < +\infty$, $\int_{\mathbb{R}} t\, d\rho(t) = 0$, and $G_\sigma(z) = h'_\rho(z)$. Conversely, for any $\rho \in \mathcal{M}$ so that $\int_{\mathbb{R}} t^2\, d\rho(t) < +\infty$, $\int_{\mathbb{R}} t\, d\rho(t) = 0$, there exists a unique $\sigma \in \mathcal{M}_d$ so that $G_\sigma(z) = h'_\rho(z)$.*

[1]It is customary in the theory of conditionally free convolution for the *second* coordinate to be chosen as the one on which $\boxplus_C$ acts as $\boxplus$, the usual free additive convolution. We have reversed this convention in our paper in order to emphasize the connection with $\boxplus_B$.

Proof. We mostly follow [1, Chapter 3]. Let us first observe that for a given positive finite measure ρ, we have

$$\int_{\mathbb{R}} t\,d\rho(t) = \lim_{y\to+\infty} iy[iyG_\rho(iy) - \rho(\mathbb{R})],$$

and

$$\int_{\mathbb{R}} t^2\,d\rho(t) = \lim_{y\to+\infty} iy\left[(iy)^2G_\rho(iy) - iy\rho(\mathbb{R}) - \int_{\mathbb{R}} t\,d\rho(t)\right],$$

provided *both* these numbers exist (these results are proved in [1, Theorem 3.2.1].) In particular, for a probability ρ, we have

$$\lim_{y\to+\infty}((iy)^4G_\rho(iy)^2 - 2(iy)^3G_\rho(iy) + (iy)^2) = \left(\int_{\mathbb{R}} t\,d\rho(t)\right)^2. \tag{11}$$

Using the monotone convergence theorem, we obtain

$$\begin{aligned}
\lim_{y\to+\infty}((iy)^2 - 2(iy)^3G_\rho(iy) - (iy)^4G_\rho'(iy)) &= \lim_{y\to+\infty}\int_{\mathbb{R}} (iy)^2 - 2\frac{(iy)^3}{iy-t} + \frac{(iy)^4}{(iy-t)^2}\,d\rho(t)\\
&= \lim_{y\to+\infty}\int_{\mathbb{R}} \frac{(ity)^2}{(iy-t)^2}\,d\rho(t)\\
&= \int_{\mathbb{R}} t^2\,d\rho(t). \qquad (12)
\end{aligned}$$

Combining (11) and (12), we get

$$\begin{aligned}
\lim_{y\to+\infty}(iy)^4(-G_\rho'(iy) - G_\rho(iy)^2) &= \lim_{y\to+\infty}(iy)^2 - 2(iy)^3G_\rho(iy) - (iy)^4G_\rho'(iy) - [yi(iyG_\rho(iy)-1)]^2\\
&= \int_{\mathbb{R}} t^2\,d\rho(t) - \left(\int_{\mathbb{R}} t\,d\rho(t)\right)^2 \qquad (13)
\end{aligned}$$

Also, by using Remark 20, we obtain that $h_\rho(z) = a + \int_{\mathbb{R}} \frac{1+tz}{t-z}\,d\tau(t)$. Under the hypotheses that the first moment of ρ is zero and the second moment of ρ is finite, we claim that τ must also have finite second moment. Indeed, assume this is not the case. According to Section 1 of [1, Chapter 3], there is a sequence $y_n \to +\infty$ so that $\lim_{n\to\infty}|y_nh_\rho(iy_n)| = \infty$. But, using the fact that the first moment of ρ is zero,

$$\lim_{n\to\infty}|y_nh_\rho(iy_n)| = \lim_{n\to\infty} y_n\left|\frac{iy_n - (iy_n)^2G_\rho(iy_n)}{iy_nG_\rho(iy_n)}\right| = \int_{\mathbb{R}} t^2\,d\rho(t) < \infty,$$

providing a contradiction.

Observing that $h_\rho'(z) = \int_{\mathbb{R}} \frac{1+t^2}{(t-z)^2}\,d\tau(t)$, we can apply the same methods as before to argue that

$$\lim_{y\to+\infty}(iy)^2h_\rho'(iy) = \int_{\mathbb{R}} 1 + t^2\,d\tau(t). \tag{14}$$

On the other hand,

$$\begin{aligned}
\lim_{y\to+\infty}(iy)^2h_\rho'(iy) &= \lim_{y\to+\infty}(iy)^2(F_\rho'(iy) - 1)\\
&= \lim_{y\to+\infty}\frac{(iy)^4(-G_\rho'(iy) - G_\rho(iy)^2)}{(iy)^2G_\rho(iy)^2}\\
&= \frac{\int_{\mathbb{R}} t^2\,d\rho(t) - \left(\int_{\mathbb{R}} t\,d\rho(t)\right)^2}{1} = \int_{\mathbb{R}} t^2\,d\rho(t) < \infty. \qquad (15)
\end{aligned}$$

Thus, $d\sigma_0(t) := (1+t^2)d\tau(t)$ is a finite measure on the real line, and by our previous remark, $h_\rho'(z) = -G_{\sigma_0}'(z) = G_\sigma(z)$, where $\sigma = \sigma_0' \in \mathcal{M}_d$.

Now proving the converse statement is easier. Indeed, if $\sigma \in \mathcal{M}_d$, then there exists at least one positive finite measure σ_0 so that $\sigma_0' = \sigma$ in distribution. Imposing the condition of finiteness specifies this measure uniquely. So we can define $\rho_a \in \mathcal{M}$ by $F_{\rho_a}(z) = a + z - G_{\sigma_0}(z)$. It follows

easily from the previous computations that in order for ρ_a to have first moment zero, it is required that $a = 0$. Similarily, the second moment of ρ_0 will be $\lim_{y\to+\infty} iyG_{\sigma_0}(iy) = \sigma_0(\mathbb{R})$. We will take $\rho = \rho_0$. Details are left to the reader as an exercise. $\square$

In our proof we will need the following result [2, Corollary 4].

Proposition 21. *Let $(\mu_1, \rho_1), (\mu_2, \rho_2) \in \mathcal{M} \times \mathcal{M}$, and denote $(\mu_3, \rho_3) = (\mu_1, \rho_1) \boxplus_C (\mu_2, \rho_2)$. Then*

$$h_{\rho_3}(z) = h_{\rho_1}(\omega_1(z)) + h_{\rho_2}(\omega_2(z)), \quad z \in \mathbb{C}^+,$$

where ω_j is the subordination function corresponding to μ_j, provided by Theorem 11 ($j \in \{1, 2\}$.)

The following theorem answers an open question mentioned in the introduction of [6].

Theorem 22. *Consider pairs $(\mu_j, \rho_j) \in \mathcal{M}\times\mathcal{M}$ so that $\int_{\mathbb{R}} t^2\, d\rho_j(t) < +\infty$, $\int_{\mathbb{R}} t\, d\rho_j(t) = 0$, $j \in \{1, 2\}$. Let $\sigma_j \in \mathcal{M}_d$ be so that $h'_{\rho_j}(z) = G_{\sigma_j}(z)$, $z \in \mathbb{C}^+$ (the existence and uniqueness of σ_j is guaranteed by Lemma 20.) Denote $(\mu_3, \rho_3) = (\mu_1, \rho_1)\boxplus_C(\mu_2, \rho_2)$ Then ρ_3 satisfies the conditions $\int_{\mathbb{R}} t^2\, d\rho_3(t) < +\infty$, $\int_{\mathbb{R}} t\, d\rho_3(t) = 0$. Moreover, if we let $(\mu_3, \sigma_3) = (\mu_1, \sigma_1) \boxplus_B (\mu_2, \sigma_2)$, then $h'_{\rho_3}(z) = G_{\sigma_3}(z)$, $z \in \mathbb{C}^+$.*

Proof. We assume first that $(\mu_1, \rho_1), (\mu_2, \rho_2)$ have all compact support. It follows then that σ_1, σ_2 also have compact support (as described in Definition 9.) Propositions 18 and 21 guarantee then that

$$g_{\sigma_3}(z) = g_{\sigma_1}(\omega_1(z))\omega_1'(z) + g_{\sigma_2}(\omega_2(z))\omega_2'(z) = (h_{\rho_1}(\omega_1(z)) + h_{\rho_2}(\omega_2(z)))' = h'_{\rho_3}(z).$$

Lemma 20 concludes the proof for the case of compactly supported measures. The general case follows by using the denseness of the set of probabilities with compact support in the space of all probabilities and Lemma 20.

$\square$

Remark 23. (1) We observe immediately that one can generalize Theorem 15 to arbitrary pairs of probabilities (μ_j, ρ_j), but at the cost of losing a significant analytic object on the second coordinate in the world of type B distributions. Indeed, relation (b) in Proposition 18, in which g_{ν_j} is replaced by h'_{ρ_j}, is easily seen to be stable when we consider weak limits $\rho_j^{(n)} \to \rho_j$.

(2) The theorem above together with the results of Krysztek [12] and of Wang [25] on conditionally free infinite divisibility and c-free limit theorems gives a complete characterization of infinite divisibility for pairs in $\mathcal{M} \times \mathcal{M}_d$.

(3) We can also conclude from the above theorem and [2, Proposition 6] the existence of the type B analogue of the Nica-Speicher partial semigroup with respect to free additive convolution. Indeed, if we denote $(\mu_t, \rho_t) = (\mu, \rho)^{\boxplus_C t}$, it follows easily from Theorem 22 and the corresponding formula $h_{\rho_t}(z) = th_\rho(\omega_t(z))$ that $(\mu_t, \sigma_t) = (\mu, \sigma)^{\boxplus_B t}$ exists for $t \geq 1$ and is defined by

$$\mu_t = \mu^{\boxplus t}, \quad g_{\sigma_t}(z) = tg_\sigma(\omega_t(z))\omega_t'(z), \tag{16}$$

for all $(\mu, \sigma) \in \mathcal{M} \times \mathcal{M}_d$. Here the function ω_t is the subordination function corresponding to the semigroup $\mu^{\boxplus t}$, provided by [3]: $F_{\mu^{\boxplus t}}(z) = F_\mu(\omega_t(z))$. We observe immediately that equation (16) allows one to extend $(\mu, \sigma)^{\boxplus_B t}$ to pairs $(\mu, \sigma) \in \mathcal{M} \times \mathcal{M}_0$.

Theorem 22 suggests a more restricted space of distributions of type B that is stable under $\boxplus_B$. Consider $f: \mathbb{R} \to \mathbb{R}$ to be a (bounded) function of bounded variation, so that $f'(t)dt = df(t)$ is a signed finite measure. We shall define ν by $g_\nu(z) = \int_{\mathbb{R}} \frac{1}{z-t} f''(t)\, dt$. The second derivative of f is taken in the distributional sense[2], so that $g_\nu(z) = \int_{\mathbb{R}} \frac{2}{(z-t)^3} f(t)\, dt$. We observe that one can recover the distribution ν from the boundary values of g_ν in an appropriate topology. For details we refer to [8] (see also [13].) Denote $\mathcal{M}_2$ the subset of $\mathcal{M}_0$ formed by functionals ν defined this way.

[2]In the most general case, one should not understand the expression $\int_{\mathbb{R}} \frac{1}{z-t} f''(t)\, dt$ as an integral, but as a linear functional applied to $t \mapsto (z-t)^{-1}$, as in the definition of $\mathcal{M}_0$. The notation $\nu\left[\frac{1}{z-t}\right]$ would be more appropriate; however, for our purposes and methods, we believe our notation to be more suggestive.

Theorem 24. *The set $\mathcal{M}\times\mathcal{M}_2$ is stable under the operation $\boxplus_B$. Moreover, for any $(\mu_j,\nu_j)\in\mathcal{M}\times\mathcal{M}_2$, $j\in\{1,2\}$, with the notation $(\mu_3,\nu_3)=(\mu_1,\nu_1)\boxplus_B(\mu_2,\nu_2)$ we have $\mu_3=\mu_1\boxplus\mu_2$ and $g_{\nu_3}(z)=g_{\nu_1}(\omega_1(z))\omega_1'(z)+g_{\nu_2}(\omega_2(z))\omega_2'(z)$, $z\in\mathbb{C}^+$, where ω_j are the subordination functions corresponding to μ_j, $j\in\{1,2\}$.*

Proof. The result follows quite easily from Proposition 18. Indeed, in order to prove our theorem, we need to show that, given f_1,f_2 generating ν_1,ν_2 as above, the formula $g_{\nu_1}(\omega_1(z))\omega_1'(z)+g_{\nu_2}(\omega_2(z))\omega_2'(z)$ has the form $g_{\nu_3}(z)$ for $\nu_3=f_3''$ in the sense of distributions, f_3 being a real function with finite variation. We first argue in the case when ν_1,ν_2 have compact support.

Consider the primitive $H_{\nu_1}(z)$ of $g_{\nu_1}(z)$ which satisfies the condition $\lim_{z\to\infty}H_{\nu_1}(z)=0$. We shall prove that $H_{\nu_1}(z)=\int_{\mathbb{R}}\frac{1}{z-t}\,df_1(t)$, $z\in\mathbb{C}^+$. Indeed, since $H_{\nu_1}'(z)=g_{\nu_1}(z)$, the uniqueness of the primitive under the hypothesis regarding the behaviour of H_{ν_1} at infinity, the definition of distributional derivative and the relations

$$\left[\int_{\mathbb{R}}\frac{1}{z-t}\,df_1(t)\right]'=-\int_{\mathbb{R}}\frac{1}{(z-t)^2}\,df_1(t)=-\int_{\mathbb{R}}\left(\frac{1}{z-t}\right)'\,df_1(t)=\int_{\mathbb{R}}\frac{1}{z-t}f_1''(t)\,dt=g_{\nu_1}(z),$$

complete the proof of our statement.

Let σ be a monotone function on the real line with finite variance. Since the function $z\mapsto\int_{\mathbb{R}}\frac{1}{\omega_1(z)-t}\,d\sigma(t)$ maps $\mathbb{C}^+$ into the lower half-plane and $\lim_{y\to+\infty}iy\int_{\mathbb{R}}\frac{1}{\omega_1(iy)-t}\,d\sigma(t)=1$, it follows from Section 1 of [1, Chapter 3] that for any such σ and ω_1 as in Theorem 11, there exists another monotone function τ with finite variance on $\mathbb{R}$ so that $\int_{\mathbb{R}}\frac{1}{\omega_1(z)-t}\,d\sigma(t)=\int_{\mathbb{R}}\frac{1}{z-t}\,d\tau(t)$, $z\in\mathbb{C}^+$. Moreover, the variance of the two functions coincides (i.e. the total mass of the real line is the same under both measures $d\sigma(t)$ and $d\tau(t)$.) Since the bounded function f_1 takes real values and has finite variance, it can be written as the difference of two positive nondecreasing functions f_1^+ and f_1^-; applying the previous observation separately to f_1^+ and f_1^- guarantees the existence of a real valued function $\tilde{f}_1$ with finite variation so that $H_{\nu_1}(\omega_1(z))=\int_{\mathbb{R}}\frac{1}{z-t}\,d\tilde{f}_1(t)$, $z\in\mathbb{C}^+$. As $g_{\nu_1}(\omega_1(z))\omega_1'(z)=H_{\nu_1}'(\omega_1(z))\omega_1'(z)$, we conclude that there exists $\tilde{\nu}_1=\tilde{f}_1''\in\mathcal{M}_2$ so that $g_{\tilde{\nu}_1}(z)=g_{\nu_1}(\omega_1(z))\omega_1'(z)$. By the same method we find $\tilde{\nu}_2=\tilde{f}_2''\in\mathcal{M}_2$ so that $g_{\tilde{\nu}_2}(z)=g_{\nu_2}(\omega_2(z))\omega_2'(z)$. It follows now easily that $f_3=\tilde{f}_1+\tilde{f}_2$ provides a $\nu_3=f_3''\in\mathcal{M}_2$ so that $g_{\nu_3}(z)=g_{\nu_1}(\omega_1(z))\omega_1'(z)+g_{\nu_2}(\omega_2(z))\omega_2'(z)$, $z\in\mathbb{C}^+$, and so, by Proposition 18, $(\mu_3,\nu_3)=(\mu_1,\nu_2)\boxplus_B(\mu_2,\nu_2)$. This holds for any $\nu_1,\nu_2\in\mathcal{M}_2$ with compact support.

The general case follows by approximating $df_j(t)$ with compactly supported measures $df_j^{(n)}(t)$, $j\in\{1,2\}$. ∎

For convenience, we shall collect in the corollary below the three spaces of distributions that have been shown to be stable under free additive convolution of type B. Of course, we do not claim these are all possible choices of such spaces, but only the ones that we consider of particular importance for the puropses of this paper.

Corollary 25. *The spaces $\mathcal{M}\times\mathcal{M}_0$, $\mathcal{M}\times\mathcal{M}_2$ and $\mathcal{M}\times\mathcal{M}_d$ are stable under the operation $\boxplus_B$ of free additive convolution of type B, where $\mathcal{M}_0$ is as in Definition 9, $\mathcal{M}_2$ consists of all $\nu\in\mathcal{M}_0$ with the property that there exists a function $f\colon\mathbb{R}\to\mathbb{R}$ of bounded variation so that $\nu[(z-t)^{-1}]=\int f(t)(z-t)^{-3}\,dt$, and $\mathcal{M}_d$ consists of all $\nu\in\mathcal{M}_2$ so that if $\nu[(z-t)^{-1}]=\int f(t)(z-t)^{-3}\,dt$, then f is nondecreasing. The space $\mathcal{M}$ is the space of all Borel probability measures on the real line.*

3.3. **Infinitesimal freeness.** We establish next the appropriate analytic framework for the correspondence bewtween type B free additive convolution and the infinitesimal freeness introduced in the previous section. Consider a path $\gamma\colon[0,1]\to\mathcal{M}$. Each probability $\gamma(t)$ has a unique nondecreasing distribution function f_t; following the above notations, we write $d\gamma(t)(x)=f_t'(x)dx$. We shall say that the path γ is differentiable if $\lim_{t\to t_0}\frac{f_t-f_{t_0}}{t-t_0}$ is a (bounded) function of bounded variation for any $t_0\in[0,1]$, where the limit is taken in the norm topology. We have then $\int(z-x)^{-n}\,d\partial_tf_t(x)=\partial_t\int(z-x)^{-n}\,df_t(x)$, $n\in\mathbb{N}$, $\Im z\neq0$.

One can easily observe that differences of probability measures belong to $\mathcal{M}_2$. Thus, it follows from the above that $\gamma'(t) \in \mathcal{M}_2$. We exploit this observation in the following result, which generalizes Proposition 17 to distributions which need not have moments:

Theorem 26. *Assume that the functions $\gamma_j \colon [0,1] \to \mathcal{M}$ are differentiable on $(0,1)$ and γ_j' extend continuously to $[0,1]$, $j \in \{1,2\}$. Then $(\gamma_1(t), \gamma_1'(t)) \boxplus_B (\gamma_2(t), \gamma_2'(t)) = (\gamma_1(t) \boxplus \gamma_2(t), \frac{d}{dt}(\gamma_1(t) \boxplus \gamma_2(t)))$, for all $t \in [0,1]$.*

Proof. Let us derivate in the subordination formula:

$$\partial_t G_{\gamma_1(t)\boxplus\gamma_2(t)}(z) = G_{\gamma_j'(t)}(\omega_j^t(z)) + G'_{\gamma_j(t)}(\omega_j^t(z))\partial_t \omega_j^t(z),$$

for any $t \in (0,1)$, $z \in \mathbb{C}^+$. (Here ω_j^t is the subordination function provided by Theorem 11 corresponding to $\gamma_j(t)$.) Part (2) of Theorem 11 implies

$$\partial_t \omega_1^t(z) + \partial_t \omega_2^t(z) = \frac{-\partial_t G_{\gamma_1(t)\boxplus\gamma_2(t)}(z)}{G_{\gamma_1(t)\boxplus\gamma_2(t)}(z)^2}.$$

Combining these two relations gives

$$\frac{\partial_t G_{\gamma_1(t)\boxplus\gamma_2(t)}(z) - G_{\gamma_1'(t)}(\omega_1^t(z))}{G'_{\gamma_1(t)}(\omega_1^t(z))} + \frac{\partial_t G_{\gamma_1(t)\boxplus\gamma_2(t)}(z) - G_{\gamma_2'(t)}(\omega_2^t(z))}{G'_{\gamma_2(t)}(\omega_2^t(z))} = \frac{-\partial_t G_{\gamma_1(t)\boxplus\gamma_2(t)}(z)}{G_{\gamma_1(t)\boxplus\gamma_2(t)}(z)^2}.$$

We multiply the two right-hand terms by $(\omega_1^t)'(z)$ and $(\omega_2^t)'(z)$, respectively and use Theorem 11 to get

$$\frac{\partial_t G_{\gamma_1(t)\boxplus\gamma_2(t)}(z)\left[(\omega_1^t)'(z) + (\omega_2^t)'(z)\right] - G_{\gamma_1'(t)}(\omega_1^t(z))(\omega_1^t)'(z) - G_{\gamma_2'(t)}(\omega_2^t(z))(\omega_2^t)'(z)}{G'_{\gamma_1(t)\boxplus\gamma_2(t)}(z)} = \frac{-\partial_t G_{\gamma_1(t)\boxplus\gamma_2(t)}(z)}{G_{\gamma_1(t)\boxplus\gamma_2(t)}(z)^2}.$$

We multiply both sides of the equality by $G'_{\gamma_1(t)\boxplus\gamma_2(t)}(z)$:

$$\partial_t G_{\gamma_1(t)\boxplus\gamma_2(t)}(z)\left[(\omega_1^t)'(z) + (\omega_2^t)'(z)\right] - G_{\gamma_1'(t)}(\omega_1^t(z))(\omega_1^t)'(z) - G_{\gamma_2'(t)}(\omega_2^t(z))(\omega_2^t)'(z) =$$
$$\partial_t G_{\gamma_1(t)\boxplus\gamma_2(t)}(z) F'_{\gamma_1(t)\boxplus\gamma_2(t)}(z).$$

Formula (2) of Theorem 11 together with the equation above assures us that

$$\partial_t G_{\gamma_1(t)\boxplus\gamma_2(t)}(z) = G_{\gamma_1'(t)}(\omega_1^t(z))(\omega_1^t)'(z) + G_{\gamma_2'(t)}(\omega_2^t(z))(\omega_2^t)'(z),$$

so an application of Theorem 24 concludes the proof. □

An obvious consequence of the above theorem is the following

Corollary 27. *Let $\mu_j, \nu_j \in \mathcal{M}$, $j \in \{1,2\}$, and denote $\mu_j^t = t\nu_j + (1-t)\mu_j$. Then $(\mu_1^t, \nu_1 - \mu_1) \boxplus_B (\mu_2^t, \nu_2 - \mu_2) = (\mu_1 \boxplus \mu_2, \frac{d}{dt}(\mu_1^t \boxplus \mu_2^t))$.*

In the following two sections we shall apply Theorem 26 and the corollary above to certain explicit type B analogues of some important distributions in free probability.

4. Some limit laws

We shall start with a discussion of the stable laws, first identified in the free context by Bercovici and Voiculescu. Recall [4, Section 7] that two probability measures μ, ν on the real line are said to have the same type if there are $s > 0, b \in \mathbb{R}$ so that $\nu(A) = \mu(sA + b)$ for any borel set $A \subseteq \mathbb{R}$. We will say that μ is stable relative to free additive convolution if $\nu \boxplus \nu'$ has the type of μ whenever ν and ν' have the type of μ.

We shall define the Voiculescu transform ϕ_ν of the probability measure ν by $\phi_\nu(z) = F_\nu^{-1}(z) - z$, where z belongs to a truncated Stolz angle at infinity. Its main property, shown in [4, Corollary 5.8], is that $\phi_{\nu\boxplus\mu}(z) = \phi_\nu(z) + \phi_\mu(z)$ for z in the common domain of the two functions[3]. It is

[3]We remind the reader that the Voiculescu transform is related to the R-transform via the formula $\phi_\nu(z) = R_\nu(1/z)$.

an easy consequence of the definition of G and ϕ that $G_\nu(z) = sG_\mu(sz+b)$ and $\phi_\nu(z) = \frac{1}{s}[\phi_\mu(sz) - b]$ whenever $\nu(A) = \mu(sA+b)$ for any Borel set $A \subseteq \mathbb{R}$.

In [4, Theorem 7.5] the authors provide a complete list of the analytic functions on $\mathbb{C}^+$ which are Voiculescu transforms of stable laws relative to free additive convolution. We recall here the list for the convenience of the reader:

(1) $\phi(z) = a$, $a \in \mathbb{R}$;
(2) $\phi(z) = a + ib$, $a \in \mathbb{R}, b < 0$;
(3) $\phi(z) = a + bz^{1-\alpha}$, $a \in \mathbb{R}, \alpha \in (1, 2]$, $\arg b \in [(\alpha - 2)\pi, 0]$;
(4) $\phi(z) = a + bz^{1-\alpha}$, $a \in \mathbb{R}, \alpha \in (0, 1)$, $\arg b \in [\pi, (1 + \alpha)\pi]$;
(5) $\phi(z) = a + b \log z$, $a \in \mathbb{C}^- \cup \mathbb{R}$, $b < 0$.

(The power and log functions are defined via their principal branches; thus, log maps the upper half-plane into $\mathbb{R} + i(0, \pi)$.)

Remark 28. We observe that if we identify constants $s(t) > 0, b(t) \in \mathbb{R}$ so that $t\phi_\mu(z) = \phi_{\mu^{\boxplus t}}(z) = \frac{1}{s(t)}[\phi_\mu(s(t)z) - b(t)]$, then the following formulas for $s(t)$ and $b(t)$ correspond to the five cases above:

Cases 1 and 2: $s(t) = 1/t, b(t) = 0$ for all $t > 0$;
Cases 3 and 4: $s(t) = t^{-1/\alpha}, b(t) = a(1 - t^{1-\frac{1}{\alpha}})$ for all $t > 0$;
Case 5: $s(t) = 1/t, b(t) = b \log t$ for all $t > 0$.

Thus, without loss of generality, we will restrict ourselves to the case when $a = 0$; this simply corresponds to translating our distributions a units, or, equivalently, convolving with either δ_{-a}, in Cases 2-4, or with $\delta_{-\Re a} \boxplus \frac{1}{\pi}\frac{-\Im a}{x^2+(\Im a)^2}dx$, in Case 5.

Stable distributions can be obtained as limits of special triangular arrays: consider a type A noncommutative probability space $(\mathcal{A}, \varphi)$; if $X_1, X_2, X_3, \dots$ are free identically distributed random variables in $(\mathcal{A}, \varphi)$, if $S_{n,\alpha} = \dfrac{X_1 + X_2 + \cdots + X_n - b_{n,\alpha}}{n^{\frac{1}{\alpha}}}$ and $\lim_{n\to\infty} \mu_{S_{n,\alpha}}$ exists, then its Voiculescu transform is of one of the five forms listed above (we denote by μ_Y the distribution of Y with respect to φ.) It is known that, except for the first case (corresponding to Dirac measures δ_a,) the numbers α from the expression of $S_{n,\alpha}$ and from the exponent in the expression of ϕ are the same; thus, in particular, $\alpha = 1$, corresponding to Case 2, provides the Cauchy distribution, and $\alpha = 2$, covered by Case 3, gives us the semicircular law (the free central limit). The last case is remarkable among all others. Even though it corresponds to $\alpha = 1$, as the Cauchy distribution, the variables X_j are 'uncentered': it is impossible to obtain a limit in this case if we try to take $b_{n,\alpha} = 0$. For details, we refer to the work of Pata [17].

Let us consider first Cases 3-4. We let μ be so that $\phi_\mu(z) = bz^{1-\alpha}$. If X_j, $j \in \mathbb{N}$, are free (with respect to a state φ,) possibly unbounded, selfadjoint random variables distributed according to μ, then we observe that $S^j_{n,\alpha} = n^{-\frac{1}{\alpha}}(X_{nj+1} + X_{nj+2} + \cdots + X_{n(j+1)})$, $j \in \mathbb{N}$, are free, identically distributed, and, as observed above, tend in distribution also to μ when n tends to infinity. Thus, not surprisingly,

$$S^1_{n,\alpha} + \cdots + S^q_{n,\alpha} = \frac{X_1 + X_2 + \cdots + X_{qn}}{n^{\frac{1}{\alpha}}} = q^{\frac{1}{\alpha}}\frac{X_1 + X_1 + \cdots + X_r}{r^{\frac{1}{\alpha}}} = q^{\frac{1}{\alpha}} S_{r,\alpha}.$$

Letting n (and hence r) tend to infinity, we obtain the obvious relation $\mu^{\boxplus q}(A) = \mu(q^{\frac{1}{\alpha}}A)$, for any Borel set $A \subseteq \mathbb{R}$. But then,

$$\begin{aligned}
\partial_q \int_{\mathbb{R}} f_z(x)\, d\mu^{\boxplus q}(x) = \partial_q G_{\mu^{\boxplus q}}(z) &= \lim_{r\to\infty} \partial_q \varphi\left([z - q^{\frac{1}{\alpha}}S_{r,\alpha}]^{-1}\right) \\
&= \lim_{r\to\infty} \varphi\left([z - q^{\frac{1}{\alpha}}S_{r,\alpha}]^{-1}\frac{1}{\alpha}q^{\frac{1}{\alpha}-1}S_{r,\alpha}[z - q^{\frac{1}{\alpha}}S_{r,\alpha}]^{-1}\right) \\
&= \frac{1}{\alpha q}\lim_{r\to\infty} \varphi\left([z - q^{\frac{1}{\alpha}}S_{r,\alpha}]^{-1}q^{\frac{1}{\alpha}}S_{r,\alpha}[z - q^{\frac{1}{\alpha}}S_{r,\alpha}]^{-1}\right),
\end{aligned}$$

where for any fixed $z \in \mathbb{C}^+$, we have denoted $f_z(x) = (z-x)^{-1}, x \in \mathbb{R}$. It is very easy to see that this equality holds also in Cases 1 and 2 (with $\alpha = 1$.) It is less trivial to observe this for Case 5, and we will give a proof below.

Assume that, in a context similar to the one described above for Cases 1-4, the random variables $X_j, j \in \mathbb{N}$ are distributed so that $\phi_\mu(z) = b \log z$. We let $S_{n,1}^j = n^{-1}(X_{nj+1} + X_{nj+2} + \cdots + X_{n(j+1)}) - b\log n$. Then

$$\begin{aligned} S_{n,1}^1 + \cdots + S_{n,1}^q &= \frac{X_1 + X_2 + \cdots + X_{qn}}{n} - qb\log n \\ &= q\left(\frac{X_1 + X_1 + \cdots + X_r}{r} - b\log r + b\log(qn) - b\log n\right) \\ &= q(S_{r,1} + b\log q). \end{aligned}$$

(We observe once again that $\phi_{\mu_{S_{n,1}}}(z) = n^{-1}\phi_{\mu_{X_1+\cdots+X_n}}(nz) - b\log n = \phi_\mu(nz) - b\log n = b\log(nz) - b\log n = b\log z = \phi_\mu(z)$, so the translation by $b\log n$ *is* necessary.) As in the previous four cases, with $z \in \mathbb{C}^+$ and $f_z(x) = (z-x)^{-1}, x \in \mathbb{R}$,

$$\begin{aligned} \partial_q \int_{\mathbb{R}} f_z(x)\, d\mu^{\boxplus q}(x) &= \partial_q G_{\mu^{\boxplus q}}(z) \\ &= \lim_{r\to\infty} \partial_q \varphi\left([z - q(S_{r,1} + b\log q)]^{-1}\right) \\ &= \lim_{r\to\infty} \varphi\left([z - q(S_{r,1} + b\log q)]^{-1}(S_{r,1} + b\log q + b)[z - q(S_{r,1} + b\log q)]^{-1}\right) \\ &= \frac{1}{q}\lim_{r\to\infty} \varphi\left([z - q(S_{r,1} + b\log q)]^{-1} q(S_{r,1} + b\log q)[z - q(S_{r,1} + b\log q)]^{-1}\right) \\ &\quad + \frac{1}{q}\lim_{r\to\infty} \varphi\left([z - q(S_{r,1} + b\log q)]^{-1} qb[z - q(S_{r,1} + b\log q)]^{-1}\right). \end{aligned}$$

Let us now consider the setup provided by Example 5 for a type B probability space. Corresponding to Cases 1-4 above, and with the same notations, let

$$\mathbb{S}_{n,\alpha} = n^{-1/\alpha}(\mathbb{X}_1 + \cdots + \mathbb{X}_n) = \begin{bmatrix} \frac{X_1+\cdots+X_n}{n^{\frac{1}{\alpha}}} & \frac{1}{\alpha}\frac{X_1+\cdots+X_n}{n^{\frac{1}{\alpha}}} \\ 0 & \frac{X_1+\cdots+X_n}{n^{\frac{1}{\alpha}}} \end{bmatrix} = \begin{bmatrix} S_{n,\alpha} & \frac{1}{\alpha}S_{n,\alpha} \\ 0 & S_{n,\alpha} \end{bmatrix},$$

let (μ, ν) be the type B distribution of $\mathbb{X}_j$, $j \in \mathbb{N}$, and denote $(\mu_q, \nu_q) = (\mu, \nu)^{\boxplus q}$. Thus,

$$n^{-1/\alpha}(\mathbb{X}_1 + \cdots + \mathbb{X}_{qn}) = \begin{bmatrix} q^{\frac{1}{\alpha}}\frac{X_1+\cdots+X_{qn}}{(qn)^{\frac{1}{\alpha}}} & q^{\frac{1}{\alpha}}\frac{X_1+\cdots+X_{qn}}{\alpha(qn)^{\frac{1}{\alpha}}} \\ 0 & q^{\frac{1}{\alpha}}\frac{X_1+\cdots+X_{qn}}{(qn)^{\frac{1}{\alpha}}} \end{bmatrix} = q^{1/\alpha}r^{-1/\alpha}(\mathbb{X}_1 + \cdots + \mathbb{X}_r),$$

and passing to the limit when $n \to \infty$ provides, together with Example 5, stability for the type B distribution (μ, ν). We will prove that $g_\nu(z) = \partial_q|_{q=1}G_{\mu_q}(z) = \partial_q|_{q=1}G_{\mu^{\boxplus q}}(z)$, where we remind the reader that the lower case g refers to the Cauchy transform of a distribution corresponding to a second coordinate in a type B probability space. Indeed, let us write, for $Z \in \mathcal{C}$,

$$\begin{aligned} \left[Z - q^{\frac{1}{\alpha}}\mathbb{S}_{r,\alpha}\right]^{-1} &= \begin{bmatrix} z - q^{\frac{1}{\alpha}}S_{r,\alpha} & w - \frac{1}{\alpha}q^{\frac{1}{\alpha}}S_{r,\alpha} \\ 0 & z - q^{\frac{1}{\alpha}}S_{r,\alpha} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (z - q^{\frac{1}{\alpha}}S_{r,\alpha})^{-1} & (z - q^{\frac{1}{\alpha}}S_{r,\alpha})^{-1}(w - \frac{1}{q}q^{\frac{1}{\alpha}}S_{r,\alpha})(z - q^{\frac{1}{\alpha}}S_{r,\alpha})^{-1} \\ 0 & (z - q^{\frac{1}{\alpha}}S_{n,\alpha})^{-1} \end{bmatrix} \end{aligned}$$

Applying (φ, φ) on the above, we obtain

$$(G_{\mu_q}(z), wG'_{\mu_q}(z) + g_{\nu_q}(z)) = (G_{\mu_q}(z), wG'_{\mu_q}(z) + q\partial_q G_{\mu^{\boxplus q}}(z));$$

evaluating this relation in $q = 1$ of course proves our claim.

We focus next on Case 5. For this we shall let

$$\mathbb{S}_{n,1}=n^{-1}(\mathbb{X}_1+\cdots+\mathbb{X}_n)=\left[\begin{array}{cc}\frac{X_1+\cdots+X_n}{n}-b\log n & b+\frac{X_1+\cdots+X_n}{n}-b\log n\\ 0 & \frac{X_1+\cdots+X_n}{n}-b\log n\end{array}\right]=\left[\begin{array}{cc}S_{n,1} & b+S_{n,1}\\ 0 & S_{n,1}\end{array}\right],$$

(thus, on the second coordinate we shift each X_j with a b) and leave the rest of notations/conventions as before. Then

$$n^{-1}(\mathbb{X}_1+\cdots+\mathbb{X}_{qn})=$$
$$\left[\begin{array}{cc}q\left(\frac{X_1+\cdots+X_{qn}}{qn}-b\log(qn)+b\log q\right) & q\left(b+\frac{X_1+\cdots+X_{qn}}{qn}-b\log(qn)+b\log q\right)\\ 0 & q\left(\frac{X_1+\cdots+X_{qn}}{qn}-b\log(qn)+b\log q\right)\end{array}\right]=$$
$$qr^{-1}(\mathbb{X}_1+\cdots+\mathbb{X}_r),$$

Obviously, as for the first four cases, we have

$$g_{\nu_q}(z)=q\partial_qG_{\mu^{\boxplus q}}(z).$$

Evaluating in $q=1$ provides the desired result.

It is worth mentioning that $\partial_qG_{\mu^{\boxplus q}}(z)$ can be expressed in terms of $G_{\mu^{\boxplus q}}(z)$ and its derivative with respect to z: for Cases 1-4, we have

$$\begin{aligned}\partial_qG_{\mu^{\boxplus q}}(z) &= \frac{1}{q\alpha}\int_{\mathbb{R}}\frac{x}{(z-x)^2}\,d\mu^{\boxplus q}(x)\\ &= \frac{1}{q\alpha}\int_{\mathbb{R}}xf_z'(x)\,d\mu^{\boxplus q}(x),\\ &= -\frac{1}{q\alpha}(G_{\mu^{\boxplus q}}(z)+zG'_{\mu^{\boxplus q}}(z)),\end{aligned}$$

while for Case 5,

$$\begin{aligned}\partial_qG_{\mu^{\boxplus q}}(z) &= \frac{1}{q}\int_{\mathbb{R}}\frac{x}{(z-x)^2}\,d\mu^{\boxplus q}(x)+\int_{\mathbb{R}}\frac{b}{(z-x)^2}\,d\mu^{\boxplus q}(x)\\ &= \frac{1}{q}\int_{\mathbb{R}}xf_z'(x)\,d\mu^{\boxplus q}(x)-bG'_{\mu^{\boxplus q}}(z)\\ &= -\frac{1}{q}(G_{\mu^{\boxplus q}}(z)+zG'_{\mu^{\boxplus q}}(z))-bG'_{\mu^{\boxplus q}}(z).\end{aligned}$$

These two formulae guarantee us that $(\mu^{\boxplus q},\partial_q\mu^{\boxplus q})\in\mathcal{M}\times\mathcal{M}_2$. (In Case 3, $\alpha=2$, $G_{\mu^{\boxplus q}}(z)$ satisfies a more famous equation, the complex Burgers equation - see (17) below, with μ from equation (17) taken to be δ_0.)

Remark 29. The operation $\boxplus_B$ behaves well with respect to translations. Indeed, let us consider $(\mu,\sigma),(\nu,\rho)\in\mathcal{M}\times\mathcal{M}_2$. Then a simple computation using Theorem 11 shows that for given translation $(\mu^b,\sigma^b)=(\mu(\cdot-b),\sigma(\cdot-b))$ we have $G_{\mu^b\boxplus\nu^c}(z)=G_{\mu\boxplus\nu}(z+b+c)$, so the subordination functions satisfy $\omega_1^{bc}(z)=\omega_1(z+b+c)-b$, $\omega_2^{bc}(z)=\omega_2(z+b+c)-c$, where $G_{\mu\boxplus\nu}(z)=G_\mu(\omega_1(z)),G_{\mu^b\boxplus\nu^c}(z)=G_{\mu^b}(\omega_2^{bc}(z))$, and similarly for ω_2,ν and c. So, according to Theorem 24, $g_{\sigma^b}(\omega_1^{bc}(z))(\omega_1^{bc})'(z)+g_{\rho^b}(\omega_2^{bc}(z))(\omega_2^{bc})'(z)=g_\sigma(\omega_1(z+b+c)-b+b)\omega_1'(z+b+c)+g_\rho(\omega_2(z+b+c)-c+c)\omega_2'(z+b+c)=g_\lambda(z+b+c)=g_{\lambda^{b+c}}(z)$, where we denote $(\mu,\sigma)\boxplus_B(\nu,\rho)=(\mu\boxplus\nu,\lambda)$. Thus, $\boxplus_B$ behaves well with respect to simple translations.

We can now prove the following corollary of Theorem 26:

Corollary 30. *Let $(\mathcal{A},\varphi)$ be a $*$-probability space of type A, and $X=X^*\in\mathcal{A}$ be selfadjoint random variable whose distribution μ is stable with respect to free additive convolution. Let $(\mathcal{A},\varphi,\mathcal{A},\varphi)$ be the type B noncommutative probability space obtained from $(\mathcal{A},\varphi)$ as in Example 5. Then there exists $b\in\mathbb{R}$ so that $(\mu,\nu)^{\boxplus_B q}=(\mu^{\boxplus q},\partial_t|_{t=1}(\mu^{\boxplus tq}))$, where (μ,ν) denotes the distribution of $(X,X+b)$ with respect to (φ,φ).*

Proof: The result follows from the above considerations, Remark 29 and Theorem 26.

Example 31. The above results allow us to recover Popa's Central Limit [18]. Indeed, the Cauchy transform of the centered semicircular law $d\gamma_t(x) = (2t^2\pi)^{-1}\sqrt{4t-x^2}dx$ of variance t is $G_{\gamma_t}(z) = (z-\sqrt{z^2-4t})/2t$. Differentiating with respect to t gives

$$\partial_t G_{\gamma_t}(z) = \frac{1}{t}\left(\frac{1}{\sqrt{z^2-4t}} - \frac{z-\sqrt{z^2-4t}}{2t}\right),$$

which is exactly t^{-1} times the second coordinate of the type B free central limit. It is easy to observe from this expression that $t\partial_t\gamma_t$ is nothing more than the difference of an arcsine law and a semicircular distribution. This corresponds to Case 3, $\alpha = 2$, among the stable distributions.

In the same way, one may obtain the type B corresesopndent of the Cauchy distribution (Case 2): if c_t is the 'centered' Cauchy distribution $dc_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2}$, $t>0$, then $G_{c_t}(z) = \frac{1}{z+it}$, so that

$$\partial_t G_{c_t}(z) = \frac{-it}{(z+it)^2},$$

and the corresponding measure is $\pi^{-1}\frac{xt^2}{(x^2+t^2)^2}dx$.

Case 1 (Dirac measures) is simple: the second coordinate is simply the distribution d which acts on any smooth function on $\mathbb{R}$ as $d(f) = f'(a)$. This is obviously not a measure.

The reader will notice that Corollary 30 does not claim to describe all B-freely stable laws. Indeed, due to the nature of type B random variables, it is not fully clear what a B-freely stable law should be. To start with, if (X,ξ) is a type B random variable, then $n^{-1}[(X,\xi)-(b,c)]$ has a Cauchy transform given on coordinates by

$$\left(nG_X(nz+b), w[nG_X(nz+b)]' + ng_\xi(nz+b) + \frac{c}{n}[nG_X(nz+b)]'\right),$$

where the reader will notice that we have already implicitly assumed to be in the context provided by Example 5. Indeed, taking the expectation of $\left(Z - n^{-1}[(X,\xi)-(b,c)]\right)^{-1}$ gives

$$\begin{aligned} E\left(\begin{bmatrix} z-\frac{X-b}{n} & w-\frac{\xi-c}{n} \\ 0 & z-\frac{X-b}{n}\end{bmatrix}^{-1}\right) &= E\begin{bmatrix} \left(z-\frac{X-b}{n}\right)^{-1} & -\left(z-\frac{X-b}{n}\right)^{-1}\left(w-\frac{\xi-c}{n}\right)\left(z-\frac{X-b}{n}\right)^{-1} \\ 0 & \left(z-\frac{X-b}{n}\right)^{-1}\end{bmatrix} \\ &= \begin{bmatrix} \varphi\left[\left(z-\frac{X-b}{n}\right)^{-1}\right] & -\left(w+\frac{c}{n}\right)\varphi\left[\left(z-\frac{X-b}{n}\right)^{-2}\right] \\ 0 & \varphi\left[\left(z-\frac{X-b}{n}\right)^{-1}\right]\end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & \frac{1}{n}\varphi\left[\left(z-\frac{X-b}{n}\right)^{-1}\xi\left(z-\frac{X-b}{n}\right)^{-1}\right] \\ 0 & 0\end{bmatrix}. \end{aligned}$$

While the first coordinate does not raise any issues, it is far from clear whether its interaction with the second should be allowed to include the addition of $\frac{c}{n}[nG_X(nz+b)]'$, when $c \neq b$, for example.

A way to answer this question would be to follow the one-variable results and consider limits of sums

$$(S_{n,\alpha}, s_{n,\alpha}) = \frac{(X_1,\xi_1)+(X_2,\xi_2)+\cdots+(X_n,\xi_n)-(b_{n,\alpha},c_{n,\alpha})}{n^{\frac{1}{\alpha}}},$$

where $(X_1,\xi_1),(X_2,\xi_2),(X_3,\xi_3),\ldots$ are B-free identically distributed in some type B probability space $(\mathcal{A},\varphi,\mathcal{A},\varphi)$, with selfadjoint components, and the distribution belongs to $\mathcal{M}\times\mathcal{M}_2$. Thus, we implicitly require that our probability space is of the form provided by Example 5. It is clear that

the first component tends to one of the freely stable laws described by Bercovici and Voiculescu. The second coordinate has a Cauchy transform given by

$$\varphi\left[\left(z-\frac{X_1+\cdots+X_n-b_{n,\alpha}}{n^{\frac{1}{\alpha}}}\right)^{-1}\frac{\xi_1+\cdots+\xi_n-c_{n,\alpha}}{n^{\frac{1}{\alpha}}}\left(z-\frac{X_1+\cdots+X_n-b_{n,\alpha}}{n^{\frac{1}{\alpha}}}\right)^{-1}\right].$$

One can observe that, provided $\frac{\xi_1+\cdots+\xi_n-c_{n,\alpha}}{n^{\frac{1}{\alpha}}}$ and $\frac{X_1+\cdots+X_n-b_{n,\alpha}}{n^{\frac{1}{\alpha}}}$ converge in distribution, our choice in the previous corollary corresponds to having X_j and ξ_j belong to the same domain of attraction. Under these circumstances, one can consider the problem of the stable laws answered. However, it is a different, and considerably more complicated, matter when these two variables belong to different domains of attraction, or especially when $\frac{\xi_1+\cdots+\xi_n-c_{n,\alpha}}{n^{\frac{1}{\alpha}}}$ does *not* converge in distribution, and we will not attempt to solve these problems here (the correspondence invoked in Remark 23 part 2 would be useful provided that the nature of the limit laws, not only their existence, could be identified.)

We discuss next in more detail several aspects of Popa's central limit theorem.

4.1. **The type B analogue of the heat equation.** It was shown in [24] that if $\mu\in\mathcal{M}$, then the type A free analogue of the heat equation is the complex Burgers equation:

$$\partial_t G_{\mu\boxplus\gamma_t}(z)+G_{\mu\boxplus\gamma_t}(z)\partial_z G_{\mu\boxplus\gamma_t}(z)=0,\quad z\in\mathbb{C}^+,t>0. \tag{17}$$

For distributions of type B, one can speak a priori of two versions of the heat equation, depending on whether one considers the parameter t to be a positive number (identified in $\mathcal{C}$ with the matrix having t on the diagonal and zero elsewhere), or one takes $\mathbf{t}=\begin{bmatrix} t & s\\ 0 & t\end{bmatrix}$ with $t>0,s\in\mathbb{R}$ instead. We shall see that in fact (probably not surprisingly) there is no difference between these two versions. To prove this statement, we shall consider the second case, and show it reduces to the first. Using the essential observation of Biane, Goodman and Nica that "type B freeness = type A freeness over $\mathcal{C}$" (see Corollary 7 and remarks following it) and analyticity of the correspondences $t\mapsto G_{\mu\boxplus\gamma_t}(z)$ and $z\mapsto G_{\mu\boxplus\gamma_t}(z)$, it is easy to observe that equation (17) holds when we replace z with $Z=\begin{bmatrix} z & w\\ 0 & z\end{bmatrix}\in\mathcal{C}$, $z\in\mathbb{C}^+$, and t with $\mathbf{t}$ from above. The meaning of ∂_Z is quite clear when one uses the power series formalism: if, as in introduction, $f(Z)=\sum_{n=0}^\infty A_nZ^n=(f_1(z),wf_1'(z)+f_2(z))$, $A_n=\begin{bmatrix} a_n & b_n\\ 0 & a_n\end{bmatrix}$, $f_1(z)=\sum_{n=0}^\infty a_nz^n$, $f_2(z)=\sum_{n=0}^\infty b_nz^n$, then on components

$$\partial_Zf(Z)=\sum_{n=1}^\infty nA_nZ^{n-1}=(f_1'(z),wf_1''(z)+f_2'(z)).$$

For functions (maps) $f(Z,\mathbf{t})\colon\mathcal{C}\times\mathcal{C}\to\mathcal{C}$, we have then

$$\begin{aligned} f(Z,\mathbf{t}) &= \sum_{m,n=0}^\infty A_{m,n}Z^m\mathbf{t}^n\\ &= \sum_{m,n=0}^\infty\begin{bmatrix} a_{m,n}z^mt^n & sna_{m,n}z^mt^{n-1}+wma_{m,n}z^{m-1}t^n+b_{m,n}z^mt^n\\ 0 & a_{m,n}z^mt^n\end{bmatrix}\\ &= \begin{bmatrix} f_1(z,t) & s\partial_tf_1(z,t)+w\partial_zf_1(z,t)+f_2(z,t)\\ 0 & f_1(z,t)\end{bmatrix}, \end{aligned}$$

where obviously $f_1(z,t)=\sum_{m,n=0}^\infty a_{m,n}z^mt^n$ and $f_2(z,w)=\sum_{m,n=0}^\infty b_{m,n}z^mt^n$. It is clear that $\partial_Zf(Z,\mathbf{t})=(\partial_zf_1(z,t),s\partial_z\partial_tf_1(z,t)+w\partial_z^2f_1(z,t)+\partial_zf_2(z,t))$ and similar for $\mathbf{t}$. Thus writing the complex Burgers equation for f in Z and $\mathbf{t}$ on components gives the usual complex Burgers equation for $f_1(z,t)$, while for the second one obtains

$$s(\underbrace{\partial_t^2f_1+\partial_t[f_1\partial_zf_1]}_{\mathfrak{B}_1})+w(\underbrace{\partial_z\partial_tf_1+\partial_z[f_1\partial_zf_1]}_{\mathfrak{B}_2})+\underbrace{\partial_tf_2+\partial_z(f_1f_2)}_{\mathfrak{Q}}=0.$$

Here both f_1, f_2 are functions of two variables (z,t). As f_1 satisfies the complex Burgers equation, it follows trivially that $\mathfrak{B}_1 = \mathfrak{B}_2 = 0$. Thus, the only nontrivial component is $\mathfrak{Q}$. The complex Burgers equation on $\mathcal{C}$ should thus be written as

$$\left\{ \begin{array}{rcl} \partial_t f_1(z,t) + f_1(z,t)\partial_z f_1(z,t) & = & 0 \\ \partial_t f_2(z,t) + \partial_z [f_1(z,t) f_2(z,t)] & = & 0 \end{array} \right. \tag{18}$$

What is remarkable is that the second coordinates of the variables do not appear at all in the equation. Thus, assume $(\gamma_t, \lambda_t), t \geq 0$ is the one-parameter semigroup of the type B free central limit law, and $(\mu, \sigma) \in \mathcal{M} \times \mathcal{M}_2$ is fixed. If we denote $(\gamma_t, \lambda_t) \boxplus_B (\mu, \sigma) = (\mathfrak{G}(t), \mathfrak{L}(t))$, then the type B free analogue of the heat equation is given by

$$\left\{ \begin{array}{rcl} \partial_t G_{\mathfrak{G}(t)}(z) + G_{\mathfrak{G}(t)}(z)\partial_z G_{\mathfrak{G}(t)}(z) & = & 0 \\ \partial_t g_{\mathfrak{L}(t)}(z) + \partial_z [G_{\mathfrak{G}(t)}(z) g_{\mathfrak{L}(t)}(z)] & = & 0 \end{array} \right. \tag{19}$$

with initial conditions

$$G_{\mathfrak{G}(0)}(z) = G_\mu(z), \quad g_{\mathfrak{L}(0)}(z) = g_\sigma(z).$$

It might be of interest to record the fact that if P is a one-variable analytic function defined on a neighbourhood of the closed upper half-plane in $\mathbb{C}$, then

$$\partial_t [g_{\mathfrak{L}(t)}(z) P(G_{\mathfrak{G}(t)}(z))] + \partial_z [g_{\mathfrak{L}(t)}(z) G_{\mathfrak{G}(t)}(z) P(G_{\mathfrak{G}(t)}(z))] = 0.$$

5. Some families of laws and the type B central limit theorem

As observed above, the (single variable) central limit theorem for infinitesimal free convolution is equivalent to the central limit theorem for type B free convolution. Such a theorem was obtained by M. Popa [18]. The associated "infinitesimal semicircle law" is one for which μ is the semicircule measure and $\mu'(t^n) = 0$ if n is odd and $\mu'(t^{2k}) = 2k\mu(t^{2k})$ (the moments of the difference of the arcsine and semicircular laws).

5.1. **Combinatorial interpretation of type B semicircle law.** As for the type A central limit distribution, a particular case of interest in this context among noncrossing partitions of type B is a noncrossing pairing:

Definition 32. *A* type B non-crossing pairing *of size n is a type B non-crossing partition π of $\{1, \ldots, n, -1, \ldots, -n\}$, so that either (i) all blocks of π, except for the zero block, consist of two elements and (ii) either π has no zero block, or its zero block has the form $\{i, j, -i, -j\}$.*

Clearly, n has to be even for such a non-crossing pairing to exist. In either case, the absolute value of π is a non-crossing pairing (of type A). In particular, exactly one of i, j must be even.

Lemma 33. *The map $\pi \mapsto Abs(\pi)$ is a $(k+1)$-to-one cover of the set of type A non-crossing pairings of $\{1, \ldots, 2k\}$ by type B non-crossing pairings of size $2k$. More precisely, given a type A non-crossing pairing π' and a set K which is either empty or is a block of π', there exists a unique non-crossing pairing $\rho(\pi, K)$ of type B, with $Abs(\pi) = \pi'$ and so that the zero block of π given by $\{\pm i : i \in K\}$.*

Proof. Let π' be a fixed type A non-crossing pairing of $\{1, \ldots, 2k\}$. Choose a block $\{i, j\}$ of π', and let's assume that $i < j$. Now consider π defined on $\{1, \ldots, n, -1, \ldots, -n\}$ as follows. First, $\{i, j, -i, -j\}$ is a block of π. Next, if $\{p, q\}$ is a block of π' with $i < p < q < j$, then both $\{p, q\}$ and $\{-p, -q\}$ are blocks of π. If $\{p, q\}$ is a block of π' with $p < q$, and either $q < i$, or $p > j$ or $p < i < j < q$, then $\{p, -q\}$ and $\{-p, q\}$ are both blocks of π. Alternatively, $\pi = \rho(\pi', K)$ can be described by insisting that $\pi|_{\{i+1, \ldots, j-1\}} = \pi'|_{\{i+1, \ldots, j-1\}}$, $\pi|_{\{j+1, \ldots, n, -1, \ldots, -(i-1)\}} = \pi'|_{\{j+1, \ldots, n, 1, \ldots, i-1\}}$ (which is non-crossing, since we have applied a cyclic permutation of the set underlying π') and by the condition that blocks of π are preserved by inversion. It is clear from this description that π is non-crossing. Moreover, it is clear from this description that this is the unique B non-crossing parition π with zero block $\{i, j, -i, -j\}$ and absolute value π'.

Since there are k choices of a block of π', we have constructed k type B pair partitions with absolute value π' (and all having a specified zero block). We can construct one more type B

partition, by stating that $\{i,j\},\{-i,-j\}$ are both blocks of π whenever $\{i,j\}$ is a block of π'. This partition has absolute value π and no zero block, and hence is the unique type B non-crossing pairing with this property. To conclude the proof, we set $\rho(\pi',\emptyset)=\pi$. □

Let C_n be the number of non-crossing pairings of $\{1,\dots,2n\}$ (thus C_n is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$). Let B_n be the number of non-crossing pairings of $\{1,\dots,2n\}$ having a nontrivial zero block. Then we see that $B_n = nC_n$ by Lemma 33. Thus we obtain a combinatorial interpretation for the moments of a type B semicircular law:

Proposition 34. *Let (μ,μ') be the infinitesimal law of a type B semicircular random variable. Then for all $k=0,1,\dots$, $\mu(t^{2k})=C_k$, $\mu'(t^{2k})=B_k$ and $\mu(t^{2k+1})=\mu'(t^{2k+1})=0$.*

5.2. **Families of type B semicircular variables.** Let $C(i_1,\dots,i_k)$ be the number of all non-crossing pairings of $\{1,\dots,k\}$ for which $i_p=i_q$ whenever $p\overset{\pi}{\sim}q$ (i.e. these are color-preserving partitions of $\{1,\dots,k\}$ in which the p-th digit is colored by the color i_p). Let $B(i_1,\dots,i_k;j)$ be the number of type B non-crossing pairings of $\{1,\dots,k\}$ for which $i_p=i_q$ whenever $|p|\overset{\pi}{\sim}|q|$ and for which the zero block contains j. Note that $B(i_1,\dots,i_k;j)=C(i_1,\dots,i_k)$, since $Abs(\pi)$ together with the designation of which pair in $Abs(\pi)$ comes from the zero block of π determines π fully, and since the parity of j determines uniquely whether it can be the smallest or largest element in a class of $Abs(\pi)$.

By analogy with the single-variable case, we shall call a family of type B non-commutative random variables $((X_1,\xi_1),\dots,(X_n,\xi_n))$ in a type B probability space $(A,\mathcal{V},\tau,f,\Phi)$ a type B semicircular family if its law is given by

$$\begin{aligned}\tau(X_{i_1}\cdots X_{i_k}) &= C(i_1,\dots,i_k)\\ f(X_{i_1}\cdots X_{i_{j-1}}\xi_{i_j}X_{i_{j+1}}\cdots X_{i_k}) &= B(i_1,\dots,i_k,;j).\end{aligned}$$

In particular, note that the variables $(X_1,\dots,X_n)$ form a free semicircular family.

Lemma 35. *Let $((X_1,\xi_1),\dots,(X_n,\xi_n))$ be type B non-commutative random variables as above. Then they are (type B) freely independent.*

Proof. Since the joint (type A) law of $(X_1,\dots,X_n)$ is that of a semicircular family, it follows that $X_1,\dots,X_n$ are (type A) freely independent.

Fix $i_1,\dots,i_k$. For π a non-crossing pairing of type A, let $c(\pi)=1$ if $i_j=i_{j'}$ whenever $j\overset{\pi}{\sim}j'$ and $c(\pi)=0$ otherwise. Then we may write, in view of Lemma 33

$$\begin{aligned}f(X_{i_1}\cdots X_{i_{j-1}}\xi_{i_j}X_{i_{j+1}}\cdots X_{i_k}) &= \sum_{\substack{\pi\text{ type }B\text{ non-crossing}\\ \text{having a zero block sarting at }j}} c(Abs(\pi))\\ &= \sum_{Abs(\pi)} c(Abs(\pi))\\ &= \sum_{\rho\text{ type }A\text{ non-crossing}}\ \prod_{\{a,b\}\text{ class of }\rho}\delta_{i_a=i_b}.\end{aligned}$$

Thus if we set $\kappa(X_1,\dots,\xi_j,\dots,X_p)=0$ unless $p=2$ and let $\kappa(X_i,X_j)=\kappa(\xi_i,X_j)=\kappa(X_i,\xi_j)=\delta_{i=j}$, then we have

$$f(X_{i_1}\cdots X_{i_{j-1}}\xi_{i_j}X_{i_{j+1}}\cdots X_{i_k})=\sum_{\rho\text{ type }A\text{ non-crossing}}\kappa_\rho(X_{i_1},\dots,\xi_{i_j},\dots,X_{i_k}).$$

Since $X_1,\dots,X_n$ are (type A) semicircular and free we also have a similar formula for τ:

$$\tau(X_{i_1}\cdots X_{i_k})=\sum_{\rho\text{ type }A\text{ non-crossing}}\kappa_\rho(X_{i_1},\dots,X_{i_k}).$$

It follows from formula (6.13) on p. 2292 of [6] that the (A') cumulants of the type B family $X_1,\dots,X_n$ are exactly the functionals κ that we defined above. Since κ obviously satisfies the condition that

mixed cumulants vanish, it follows that our family is indeed type B freely independent (see Theorem 6.4 on p. 2293, Proposition on p. 2298 and Corollary on p. 2300 of [6]). □

5.3. **The type B Poisson limit theorem.** We consider next the issue of the Poisson limit theorem for type B distributions. Recall from the paper of M. Popa [18] that a type B Bernoulli variable has the type B law given by

$$E((a,\xi)^n) = \Lambda A^n$$

where $\Lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 0 & \lambda_1 \end{bmatrix}$ and $A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & \alpha_1 \end{bmatrix}$. The formula is valid for all $n \geq 1$ (for $n = 0$ the correct answer is the identity in $\mathcal{C}$). Thus explicitely we have

$$E((a,\xi)^n) = \begin{bmatrix} \lambda_1\alpha_1^n & \lambda_2\alpha_1^n + n\lambda_1\alpha_1^{n-1}\alpha_2 \\ 0 & \lambda_1\alpha_1^n \end{bmatrix}.$$

Let now

$$X_t = \alpha_1 \exp\left(\frac{\alpha_2}{\alpha_1}t\right) p_t,$$

where p_t is a family of projections so that $\tau(p_t) = \lambda_1 + t\lambda_2$.[4]Then:

$$\begin{aligned} \tau(X_t^n) &= \alpha_1^n \exp\left(\frac{\alpha_2}{\alpha_1}nt\right)\tau(p_t) = \alpha_1^n \exp\left(\frac{\alpha_2}{\alpha_1}nt\right)(\lambda_1 + t\lambda_2) \\ &= \lambda_1\alpha_1^n + t(\alpha_1^n\frac{\alpha_2}{\alpha_1}n\lambda_1 + \alpha_1^n\lambda_2) + O(t^2) \\ &= \lambda_1\alpha_1^n + t(\lambda_2\alpha_1^n + n\lambda_1\alpha_1^{n-1}\alpha_2) + O(t^2), \end{aligned}$$

so that X_t has the desired infinitesimal law.

Thus, one can re-interpret Popa's type B Poisson summation theorem as essentially a direct consequence of the free Poisson summation theorem, since for each fixed t, appropriately rescaled free sums of X_t^n will have a free Poisson law with parameters $(\alpha_1\exp(t\alpha_2/\alpha_1), \lambda_1 + t\lambda_2)$ as a limit. (We remind the reader that a free Poisson law with parameters (α, λ) is the result of applying the homotethy by α to the law with with R-transform $\lambda z(1-z)^{-1}$).

We state our result in the following corollary:

Corollary 36. *Free type B Poisson laws are infinitesimals to the family of free Poisson laws with the parameters $(\alpha_1\exp(t\alpha_2/\alpha_1), \lambda_1 + t\lambda_2)$. An operator model for these laws is*

$$Y_t = \alpha_1\exp(t\alpha_2/\alpha_1)Xp_tX = X\ B_tX$$

where X is a standard semicircular and p_t are projections with $\tau(p_t) = \lambda_1 + t\lambda_2$, and $B_t = \alpha_1\exp(t\alpha_2/\alpha_1)$ is type B Bernoulli. Moreover, any product of the form

$$Z_t\ B_t\ Z_t$$

where Z_t is a type B semicircular and B_t a type B Bernoulli, is a type B Poisson.

In the spirit of the above corollary, we recall that, as shown by Nica and Speicher in [15], for an arbitrary probability measure on the real line μ, one can define the t^{th} free additive convolution power $\mu^{\boxplus t}$ for any $t \geq 1$; moreover, [15] provides also an operatorial representation of $\mu^{\boxplus t}$. If $X = X^* \in (\mathcal{A}, \varphi)$ is distributed according to μ, and $p(t) = p(t)^2 = p(t)^* \in (\mathcal{A}, \varphi)$ is a projection free from X with $\varphi(p(t)) = t^{-1}$, then $tp(t)Xp(t) \in (p(t)\mathcal{A}p(t), \frac{1}{t}\varphi)$ is distributed according to $\mu^{\boxplus t}$. Thus, in a certain sense, derivating along the path $t \mapsto \mu^{\boxplus t}$ with respect to t corresponds at an operatorial level to derivating along a path of free projectors. This observation holds in particular for the stable laws described in Corollary 30.

[4]Of course, we could modify X_t by removing higher order terms in t, so for example we could use $X_t = \alpha_1(1 + t\alpha_2/\alpha_1)p_t$.

6. A matrix model for type B freeness.

The main result of this section deals with the asymptotics of the law of a matrix X_N whose entries are semicircular variables. More precisely, we assume that $X_N = X_N^*$ is $N \times N$ and that its entries $\{X_N^{ij} : 1 \leq i, j \leq N\}$ form a free semicircular family so that the covariance of X_N^{ij} is $N^{-1/2}(1 + \delta_{i=j})$. The matrix X_N is a free probability analog of a real Gaussian random matrix (the free probability analog of a complex Gaussian random matrix differs in that the off-diagonal entries X_N^{ij}, $i < j$, are circular rather than semicircular). Note in particular that $X_N = X_N^t$ if by the latter we denote the a kind of transpose of X_N obained by switching its rows and columns.

Although X_N has in the $N \to \infty$ limit a semicircular law of variance 1, its law varies with N (in fact, we'll show that the law of X_N is seicircular of variance $(1+1/N)$ for all N). Thus one has a canonical infinitesimal law associated to X_N. Indeed, every moment of X_N admits an expansion in powers of $1/N$. Thus moments taken to order $t = 1/N$ give rise to an infinitesimal law, and the main reult of this section (Corollary 39) states that the infinitesimal law associated to the matrices X_N as $N \to \infty$ is the same as the infinitesimal law associated to a type B semcircular variable.

6.1. **Matrices with entries free creation operators.** Let $\ell(i, j, k)$ be a family of $*$-free creation operators in a non-commutative probability space (A, ψ). We thus assume that

$$\ell(i, j, k)^* \ell(i', j', k') = \delta_{i=i',\ j=j',\ k=k'} 1$$

and $\psi(w) = 0$ whenever w is a word involving $\ell(i, j, k)$'s and their adjoints that cannot be reduced to a scalar using the relation above. It is well-known (see e.g. [...]) that these two requirements completely determine the joint $*$-distribution of this family and moreover that the operators $\{\ell(i, j, k)\}_{i,j,k}$ are $*$-free.

Consider algebra $M_{N\times N}(A) = M_{N\times N}(\mathbb{C}) \otimes A$ of $N \times N$ matrices with entries from A, endowed with the state $\psi_N = \frac{1}{N} Tr \otimes \psi$. Let $L_N(k)$ be the matrix

$$L_N(k) = \frac{1}{\sqrt{N}} (\ell(i, j, k))_{1 \leq i,j \leq N}$$

and let $L_N(k)^t$ be its "transpose":

$$L_N(k)^t = \frac{1}{\sqrt{N}} (\ell(j, i, k))_{1 \leq i,j, \leq N}.$$

Lemma 37. *The elements $L_N(k)$, $L_N(k)^t$ satisfy:*

$$\begin{aligned} L_N(k)^* L_N(k') &= \delta_{k=k'} 1 \\ (L_N(k)^t)^* L_N(k')^t &= \delta_{k=k'} 1 \\ (L_N(k)^t)^* L_N(k') &= \delta_{k=k'} \frac{1}{N} \\ (L_N(k))^* L_N(k')^t &= \delta_{k=k'} \frac{1}{N}. \end{aligned}$$

Moreover, if w is a word in $\{L_N(k), L_N(k)^t : k = 1, 2, \dots\}$ and their adjoint, which cannot be reduced to a scalar using these relations, then $\psi_N(w) = 0$.

Proof. We compute the a, b-th entry of each of the matrices listed above and use the relations among $\ell(i, j, k)$ and their adjoints. For the matrix $(L_N(k)^t)^* L_N(k')$, we get as the a, b-th entry

$$\frac{1}{N} \sum_i \ell(a, i, k)^* \ell(i, b, k') = \delta_{k=k'} \delta_{a=b} \frac{1}{N}.$$

The other comutations are similar.

Finally, if w is an irreducible words, then all entries of w are either zero, or are irreducible words and thus yield zero when supplied as an argument to ψ. $\square$

6.2. **The law of X_N.**

Corollary 38. *Let $\{s(i,j,k) : 1 \leq i \leq j \leq N, k = 1,2,\dots\}$ be a free semicircular family, and let X_N be an $N \times N$ matrix whose i,j-th entry is $(1+\delta_{i=j})N^{-1/2}s(i,j,k)$ if $i \leq j$ and $s(j,i,k)$ if $i > j$. Then $\{X_N(k) : k = 1,2,\dots\}$ form a free semicircular family, and each $X_N(k)$ is semicircular with variance $(1+1/N)$. In particular, for each k, $\psi_N(X_N(k)^m) = 0$ if m is odd, and*

$$\psi_N(X_N(k)^{2n}) = C_n \left(1 + \frac{1}{N}\right)^n,$$

where C_n is the number of non-crossing pairings of $\{1,\dots,2n\}$.

Proof. Let $\hat{X}_N(k) = \frac{1}{\sqrt{2}}(L_N(k) + L_N(k)^* + L_N(k)^t + (L_N(k)^t)^*)$. Then $\hat{X}_N(k)$ is self-adjoint and its i,j-th entry, for $i \leq j$ is equal to $N^{-1/2}(\ell(i,j,k) + \ell(j,i,k) + \ell(i,j,k)^* + \ell(j,i,k)^*)/\sqrt{2}$. Since $l(i,j,k) = (\ell(i,j,k) + \ell(j,i,k))/\sqrt{2}$ is a free creation operator for $i \neq j$ and is $\sqrt{2}$ times a free creation operator for $i = j$, it follows that the joint law of the entries of $\{\hat{X}_N(k) : k = 1,2,\dots\}$ and $\{X_N(k) : k = 1,2,\dots\}$ are the same. Thus the laws of these matrices are the same as well.

$L_k = 2^{-\frac{1}{2}}(L_N(k) + L_N(k)^t)$, so that $X_N(k) = L_k + L_k^*$. Note that $L_k^* L_{k'} = (1+1/N)\delta_{k=k'}$ and $\psi_N(w) = 0$ whenever w is a word in $\{L_k, L_k^* : k = 1,2,\dots\}$ which cannot be reduced to a scalar using this relation. It then follows that each $L_k + L_k^*$ is semicircular of variance $(1+1/N)$, and that $\{L_k : k = 1,2,\dots\}$ are $*$-free. □

6.3. **Infinitesimal laws associated to X_N as $N \to \infty$.** Let $\{X_N(k) : k = 1,2,\dots\}$ be matrices as defined in Corollary 38. Thus the i,j-th entry $s(i,j,k)$ of $X_N(k)$ is a semicircular variable of variance $N^{-1/2}(1+\delta_{i=j})$ and the variables $\{s(i,j,k) : 1 \leq i,j \leq N, k = 1,2,\dots\}$ are assumed to be freely independent.

We now consider the infinitesimal law of the family $\{X_N(k) : k = 1,2,\dots\}$ as $t = 1/N$ approaches 0 (so $N \to \infty$). Thus we define μ, μ' by

$$\begin{aligned} \mu(p) &= \lim_{1/N \to 0} \psi_N(p(X_N(1),\dots,X_N(m)), \\ \mu'(p) &= \lim_{1/N \to 0} \frac{1}{1/N}(\psi_N(p(X_N(1),\dots,X_N(m)) - \mu(p)), \end{aligned}$$

for p an arbitrary non-commutative polynomial in m variables $t_1,\dots,t_m$.

Corollary 39. *Let (μ,μ') be as defined in the previous paragraph. Then the variables $t_1,\dots,t_m$ are infinitesimally free when considered with the law (μ,μ'). Furthermore, the restriction of (μ,μ') to any variable t_k yields an infinitesimal (i.e., type B) semicircular law.*

Proof. Indeed, the joint law of $\{X_N(k) : k = 1,2,\dots\}$ with respect to ψ_N is that of a free semcircular family of covariance $(1+1/N)$. The statement of the corollary now follows using Corollary 30 applied to the case of the semicircular distribution and from Proposition 16. □

As we noted in the introduction, it is necessary to start with a matrix X_N with semicircular entries; a real Gaussian random matrix will not work.

7. Multiplicative free convolution of type B

It has been shown in [18] that one can define a natural multiplicative analogue of the operation $\boxplus_B$, which we will denote by $\boxtimes_B$. Surprisingly, the main additive results have a formal multiplicative analogue, with the notable exception of Theorem 22. Unlike in the additive case, it is far from clear what would be the narrowest appropriate class of analytic objects for the second coordinate in the multiplicative case. Since proofs are identical to the ones from the previous section, we will provide only the statements of our results.

First we shall introduce several notations. For any probability measure μ, let

$$\psi_\mu(z) = \int \frac{zt}{1-zt}\, d\mu(t),$$

be its *moment generating function.* If μ is supported on the unit circle $\mathbb{T}$ in the complex plane, then ψ_μ is defined and analytic inside the unit disc $\mathbb{D}$ and takes values in the half-plane $\{z\colon \Re z \geq -1/2\}$. If μ is supported on the positive half-line $[0,+\infty)$, then ψ_μ is an analytic self-map of $\mathbb{C}\setminus[0,+\infty)$ which preserves the upper and lower half-planes and increases the argument: $\pi > \arg\psi_\mu(z) \geq \arg z$ for $0 < \arg z < \pi$. In the following we shall denote by $\mathcal{M}_{\mathbb{T}}$ the set of Borel probability measures supported on the unit circle and by $\mathcal{M}_+$ the set of Borel probability measures supported on the positive half-line. Biane [5] has shown that a subordination result holds also for free multiplicative convolution:

Theorem 40. *Let μ_1,μ_2 be two Borel probability measures, and denote by $\mu_3=\mu_1\boxtimes\mu_2$ their free multiplicative convolution.*

(1) *If $\mu_1,\mu_2\in\mathcal{M}_{\mathbb{T}}$, then there exist two analytic self-maps ω_1,ω_2 of the unit disc so that*
   (a) $|\omega_j(z)|\leq|z|$, $z\in\mathbb{D}$, $j\in\{1,2\}$;
   (b) $\psi_{\mu_j}(\omega_j(z))=\psi_{\mu_1\boxtimes\mu_2}(z)$, $z\in\mathbb{D}$, $j\in\{1,2\}$.

(2) *If $\mu_1,\mu_2\in\mathcal{M}_+$, then there exist two analytic self-maps ω_1,ω_2 of the slit complex plane $\mathbb{C}\setminus[0,+\infty)$ so that*
   (a) $\pi>\arg\omega_j(z)\geq\arg z$, $z\in\mathbb{C}^+$, $j\in\{1,2\}$;
   (b) $\psi_{\mu_j}(\omega_j(z))=\psi_{\mu_1\boxtimes\mu_2}(z)$, $z\in\mathbb{C}\setminus[0,+\infty)$, $j\in\{1,2\}$.

   Moreover, in both cases the subordination functions satisfy the following relation:

$$\frac{z\psi_{\mu_1\boxtimes\mu_2}(z)}{1+\psi_{\mu_1\boxtimes\mu_2}(z)}=\omega_1(z)\omega_2(z), \tag{20}$$

for z in the domain of $\psi_{\mu_1\boxtimes\mu_2}$.

Using this result and the work of Popa [18], one can prove the following results:

Proposition 41. *Consider two type B random variables $(a_1,\xi_1),(a_2,\xi_2)$ which are B-free and distributed according to (μ_1,ν_1) and (μ_2,ν_2), respectively. Assume all these distributions are compactly supported. Denote by (μ_3,ν_3) the distribution of $(a_1,\xi_1)(a_2,\xi_2)=(a_1a_2,a_1\xi_2+\xi_1a_2)$. Then, with the notations from Theorem 40, we have*

(a) $\mu_3=\mu_1\boxtimes\nu_1$;
(b) $\dfrac{\psi_{\nu_3}(z)}{z}=\dfrac{\psi_{\nu_1}(\omega_1(z))}{\omega_1(z)}\omega_1'(z)+\dfrac{\psi_{\nu_2}(\omega_2(z))}{\omega_2(z)}\omega_2'(z)$.

Observe that we have not specified the domain on which the functions above are defined, or the analytic nature of ν_j, at this moment. While the domains of the functions involved are rather easy to find (the unit disc for distributions supported on $\mathbb{T}$ and the slit complex plane for distributions on the positive half-line), unfortunately it is not clear what appropriate sets of distributions are stable under $\boxtimes_B$. The following analogue of Theorem 26 is the only exception we know.

Theorem 42. *Assume that the functions $\gamma_j\colon[0,1]\to\mathcal{M}_\epsilon$ ($\epsilon\in\{\mathbb{T},+\}$) are differentiable on $(0,1)$ and γ_j' extend continuously to $[0,1]$, $j\in\{1,2\}$. Then $(\gamma_1(t),\gamma_1'(t))\boxtimes_B(\gamma_2(t),\gamma_2'(t))=(\gamma_1(t)\boxtimes\gamma_2(t),\frac{d}{dt}(\gamma_1(t)\boxtimes\gamma_2(t)))$, for all $t\in[0,1]$.*

Proof. The proof is similar to the one of Theorem 26. We derivate in the subordination formula:

$$\partial_t\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)=\psi_{\gamma_j'(t)}(\omega_j^t(z))+\psi_{\gamma_j(t)}'(\omega_j^t(z))\partial_t\omega_j^t(z),$$

for $j\in\{1,2\}$, with ω_j^t denoting the subordination function provided by Theorem 40. From (20), together with the formula above, we get

$$\frac{z\partial_t\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)}{(1+\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z))^2}=\frac{\partial_t\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)-\psi_{\gamma_1'(t)}(\omega_1^t(z))}{\psi'_{\gamma_1(t)}(\omega_1^t(z))}\omega_2^t(z)+\frac{\partial_t\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)-\psi_{\gamma_2'(t)}(\omega_2^t(z))}{\psi'_{\gamma_2(t)}(\omega_2^t(z))}\omega_1^t(z).$$

Recalling that $\psi'_{\gamma_1(t)\boxtimes\gamma_2(t)}(z) = \psi'_{\gamma_j(t)}(\omega_j^t(z))(\omega_j^t)'(z)$, we may amplify the two fractions in the right hand term of the equality above by $(\omega_1^t)'(z)$ and $(\omega_2^t)'(z)$, respectively, and multiply by $\psi'_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)$ in order to get, after regrouping,

$$z\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)\left(\frac{\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)}{1+\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)}\right)' = \psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)[\omega_1^t(z)(\omega_2^t)'(z) + (\omega_1^t)'(z)\omega_2^t(z)]$$
$$-\psi_{\gamma_1'(t)}(\omega_1^t(z))(\omega_1^t)'(z)\omega_2^t(z) - \psi_{\gamma_2'(t)}(\omega_2^t(z))(\omega_2^t)'(z)\omega_1^t(z).$$

Equation (20) implies that $z\left(\frac{\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)}{1+\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)}\right)' - \omega_1^t(z)(\omega_2^t)'(z) - (\omega_1^t)'(z)\omega_2^t(z) = -\frac{\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)}{1+\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)}$. Using this relation, mutliplying the equation above by $[\omega_1^t(z)\omega_2^t(z)]^{-1}$ and applying again (20) provides

$$\frac{\partial_t\psi_{\gamma_1(t)\boxtimes\gamma_2(t)}(z)}{z} = \frac{\psi_{\gamma_1'(t)}(\omega_1^t(z))}{\omega_1^t(z)}(\omega_1^t)'(z) + \frac{\psi_{\gamma_2'(t)}(\omega_2^t(z))}{\omega_2^t(z)}(\omega_2^t)'(z).$$

The above equality holds for $z \in \mathbb{D}$ if $\epsilon = \mathbb{T}$ and for $z \in \mathbb{C}\setminus[0,+\infty)$ if $\epsilon = +$. The theorem follows now from Proposition 41. □

References

[1] Akhieser, N. I. *The classical moment problem and some related questions in analysis.* Hafner Publishing Co., New York 1965

[2] Belinschi, S. T. *C-free convolution for measures with unbounded support.* C^*-algebras in Sibiu, 1–7. Theta 2008

[3] Belinschi, S. T.; Bercovici, H. *Atoms and regularity for measures in a partially defined free convolution semigroup.* Math. Z., **248**, (2004), 665-674.

[4] Bercovici, H.; Voiculescu, D. *Free convolution of measures with unbounded support.* Indiana Univ. Math. J. Vol. 42, No 3 (1993) 733–773.

[5] Biane, Philippe. *Processes with free increments.* Math. Z. 227 (1998), no.1, 143-174.

[6] Biane, Philippe; Goodman, Frederick; Nica, Alexandru. *Non-crossing cumulants of type B.* Trans. Amer. Math. Soc. 355 (2003), no. 6, 2263–2303

[7] Bożejko, Marek; Leinert, Michael; Speicher, Roland. *Convolution and limit theorems for conditionally free random variables.* Pacific J. Math. 175 (1996), no.2, 357–388.

[8] Carmichael, Richard D.; Kamiński, Andrzej; Pilipović, Stevan. *Boundary values and convolution in ultradistribution spaces.* Series on Analysis, Applications and Computation, 1. World Scientific Publishing Co. 2007.

[9] Dumitriu, Ioana; Edelmann, Alan. *Global spectrum fluctuations for the β-Hermite and β-Laguerre ensemble via matrix models.* J. Math. Phys. **47** (2006), no. 6, 063302, 36pp.

[10] Duren, Peter L. *Theory of H^p spaces.* Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London 1970.

[11] Johansson, K. *On fluctuations of eigenvalues of random Hermitian matrices.* Duke Math. J., 91(1): 151-204, 1998.

[12] Krystek, Anna Dorota. *Infinite divisibility for the conditionally free convolution.* Infin. Dimens. Anal. Quantum Probab. Relat. Top. **10** (2007), no.4, 499–522.

[13] Luszczki, Z; Zieleźny, Z. *Distributionen der Räume D'_{L^p} als Randverteilungen Analytischer Funktionen.* Colloq. Math. Vol. VIII (1961), Fasc. 1, 125–131.

[14] Mingo, James; Speicher, Roland. *Second order freeness and fluctuations of random matrices: I. Gaussian and Wishart matrices and cyclic Fock spaces.* J. Funct. Anal., 235, 2006, pp. 226–270.

[15] Nica, Alexandru; Speicher, Roland. *On the multiplication of free N-tuples of noncommutative random variables*, Amer. J. Math. **118** (1996), 799–837.

[16] Nica, Alexandru; Speicher, Roland. *Lectures on the combinatorics of free probability.* Cambridge University Press, 2006.

[17] Pata, Vittorino. *Lévy type characterization of stable laws for free random variables.* Trans. Amer. Math. Soc. **347**, No.7, (1995), 2457–2472.

[18] Popa, Mihai V. *Limit theorems and S-transform in non-commutative probability spaces of type B.* Preprint (2007) arXiv:0709.0011.

[19] Reiner, V. *Non-crossing partitions for classical reflection groups.* Discrete Math. 177 (1997), 195–222

[20] Schwartz, Laurent. *Théorie des distributions II.* Paris 1951.

[21] Speicher, Roland. *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory.* Memoirs of the Amer. Math. Soc. 132 (1998), x+88

[22] Voiculescu, Dan. *Addition of certain non-commutative random variables.* J. Funct. Anal., 66 (1986), 323–346

[23] Voiculescu, D.; Dykema, K.; Nica, A. *Free random variables.* CRM Monograph Series, Vol. 1, AMS, 1992.

[24] Voiculescu, Dan. *The analogues of entropy and of Fisher's information measure in free probability. I.* Comm. Math. Phys. 155 (1993), no. 1, 71–92.

[25] Wang, Jiun-Chau. *Work in progress.*